

SINGULAR PERTURBATIONS OF ABSTRACT WAVE EQUATIONS

ANDREA POSILICANO

ABSTRACT. Given, on the Hilbert space \mathcal{H}_0 , the self-adjoint operator B and the skew-adjoint operators C_1 and C_2 , we consider, on the Hilbert space $\mathcal{H} \simeq D(B) \oplus \mathcal{H}_0$, the skew-adjoint operator

$$W = \begin{bmatrix} C_2 & 1 \\ -B^2 & C_1 \end{bmatrix}$$

corresponding to the abstract wave equation $\ddot{\phi} - (C_1 + C_2)\dot{\phi} = -(B^2 + C_1C_2)\phi$. Given then an auxiliary Hilbert space \mathfrak{h} and a linear map $\tau : D(B^2) \rightarrow \mathfrak{h}$ with a kernel \mathcal{K} dense in \mathcal{H}_0 , we explicitly construct skew-adjoint operators W_Θ on a Hilbert space $\mathcal{H}_\Theta \simeq D(B) \oplus \mathcal{H}_0 \oplus \mathfrak{h}$ which coincide with W on $\mathcal{N} \simeq \mathcal{K} \oplus D(B)$. The extension parameter Θ ranges over the set of positive, bounded and injective self-adjoint operators on \mathfrak{h} .

In the case $C_1 = C_2 = 0$ our construction allows a natural definition of negative (strongly) singular perturbations A_Θ of $A := -B^2$ such that the diagram

$$\begin{array}{ccc} W & \longrightarrow & W_\Theta \\ \uparrow & & \downarrow \\ A & \longrightarrow & A_\Theta \end{array}$$

is commutative.

1. INTRODUCTION

Given a negative and injective self-adjoint operator $A = -B^2$ on the Hilbert space \mathcal{H}_0 with scalar product $\langle \cdot, \cdot \rangle_0$ and corresponding norm $\| \cdot \|_0$, we consider the abstract wave equation

$$\ddot{\phi} = A\phi.$$

The Cauchy problem for such an equation is well-posed and

$$\phi(t) := \cos tB \phi_0 + B^{-1} \sin tB \dot{\phi}_0$$

is the (weak) solution with initial data $\phi_0 \in D(B)$ and $\dot{\phi}_0 \in \mathcal{H}$. More precisely, using a block matrix operator notation,

$$\begin{bmatrix} \cos tB & B^{-1} \sin tB \\ -B \sin tB & \cos tB \end{bmatrix}$$

1

defines a strongly continuous group of evolution on the Hilbert space $\mathcal{H}_1 \oplus \mathcal{H}_0$, where \mathcal{H}_1 denotes $D(B)$ endowed with the scalar product giving rise to the graph norm. It preserves the energy

$$\mathcal{E}(\phi, \dot{\phi}) := \frac{1}{2} \left(\|\dot{\phi}\|_0^2 + \|B\phi\|_0^2 \right)$$

and, in the case the Hilbert space \mathcal{H}_0 is real, constitutes a group of canonical transformations with respect to the standard symplectic form

$$\Omega((\phi_1, \dot{\phi}_1), (\phi_2, \dot{\phi}_2)) := \langle \phi_1, \dot{\phi}_2 \rangle_0 - \langle \phi_2, \dot{\phi}_1 \rangle_0.$$

Its generator is given by

$$\overset{\circ}{W} = \begin{bmatrix} 0 & 1 \\ -B^2 & 0 \end{bmatrix} : D(B^2) \oplus D(B) \subseteq \mathcal{H}_1 \oplus \mathcal{H}_0 \rightarrow \mathcal{H}_1 \oplus \mathcal{H}_0;$$

it is the Hamiltonian vector field corresponding, via Ω , to the Hamiltonian function \mathcal{E} .

From the point of view of Hamiltonian systems (with infinite degrees of freedom) a more suitable phase space is given by the space of finite energy states, i.e. the maximal domain of definition of the energy \mathcal{E} . This set is given by $D(\mathcal{E}) = \bar{\mathcal{H}}_1 \oplus \mathcal{H}_0$ where $\bar{\mathcal{H}}_1$ denotes the Hilbert space obtained by completing $D(B)$ endowed with the scalar product

$$[\phi_1, \phi_2]_1 := \langle B\phi_1, B\phi_2 \rangle_0.$$

By our injectivity hypothesis $0 \notin \sigma_{pp}(A)$, but $0 \in \sigma(A) \setminus \sigma_{pp}(A)$ is not excluded (e.g. when $A = \Delta$ and $\mathcal{H}_0 = L^2(\mathbb{R}^d)$). Thus in general $\bar{\mathcal{H}}_1$ is not contained into \mathcal{H}_0 .

It is then possible to define a new operator W which is proven to be skew-adjoint on the Hilbert space $D(\mathcal{E})$. Such an operator is nothing but the closure of $\overset{\circ}{W}$, now viewed as an operator on the larger space $D(\mathcal{E})$. By Stone's theorem W generates a strongly continuous group U^t of unitary operators which preserves the energy, which now coincides with the norm of the ambient space.

Consider now a self-adjoint operator $\hat{A} \neq A$ which is a singular perturbation of A , i.e the set $\mathcal{K} := \{\phi \in D(A) \cap D(\hat{A}) : A\phi = \hat{A}\phi\}$ is dense in \mathcal{H}_0 (see e.g. [8]). Since \mathcal{K} is closed with respect to the graph norm on $D(A)$, the linear operator $A_{\mathcal{K}}$, obtained by restricting A to the set \mathcal{K} , is a densely defined closed symmetric operator. Therefore the study of singular perturbations of A is brought back to the study of self-adjoint extensions of the symmetric operators obtained by restricting A to some dense, closed with respect to the graph norm, set. We refer to [2] and its huge list of references for the vast literature on the subject. However here we found more convenient to use the approach introduced in [11].

In the case the singular perturbation \hat{A} is negative and injective, we are interested in describing \hat{W} , the analog of W relative to \hat{A} . A natural question is:

1. Is \hat{W} a singular perturbation of W ?

Here a skew-adjoint operator \hat{W} on $D(\hat{\mathcal{E}}) \supseteq D(\mathcal{E})$ is said to be a singular perturbation of the skew-adjoint operator W on $D(\mathcal{E})$ if the set $\mathcal{N} := \{(\phi, \dot{\phi}) \in D(W) \cap D(\hat{W}) : W(\phi, \dot{\phi}) = \hat{W}(\phi, \dot{\phi})\}$ is dense in $D(\mathcal{E})$. In the case the answer to question 1 is affirmative, two other natural questions arise:

2. Is it possible to construct such singular perturbations \hat{W} without knowing \hat{A} in advance?
3. Is it possible to recover the singular perturbation \hat{A} of A from the singular perturbation \hat{W} of W ? In other words, is the following diagram commutative?

$$\begin{array}{ccc} W & \longrightarrow & \hat{W} \\ \uparrow & & \downarrow \\ A & \longrightarrow & \hat{A} \end{array}$$

Let us remark that in the case \hat{A} is a strongly singular perturbation of A , i.e. when the form domains of A and \hat{A} are different, the spaces $D(\mathcal{E})$ and $D(\hat{\mathcal{E}})$ are different, so that W and \hat{W} are defined on different Hilbert spaces. Indeed we will answer question 2 above by looking for singular perturbations with $D(\hat{\mathcal{E}}) \simeq D(\mathcal{E}) \oplus (D(A)/\mathcal{K})$. This results to be the right ansatz to give affirmative answers to questions 1 and 3.

The framework described above can be extended by considering generalized abstract wave equations of the kind

$$\ddot{\phi} - (C_1 + C_2)\dot{\phi} = (A - C_1C_2)\phi,$$

with both C_1 and C_2 skew-adjoint operators such that $A - C_1C_2$ is negative and injective. The corresponding block matrix operator is

$$\overset{\circ}{W}_g = \begin{bmatrix} C_2 & 1 \\ -B^2 & C_1 \end{bmatrix} : D(B^2) \oplus D(B) \subseteq \mathcal{H}_1 \oplus \mathcal{H}_0 \rightarrow \mathcal{H}_1 \oplus \mathcal{H}_0.$$

Then $\overset{\circ}{W}_g$ is closable, with closure W_g , as an operator on the Hilbert space $D(\mathcal{E}_C)$, the completion of $\mathcal{H}_1 \oplus \mathcal{H}_0$ with respect to the scalar product

$$\langle (\phi_1, \dot{\phi}_1), (\phi_2, \dot{\phi}_2) \rangle_{\mathcal{E}_C} := \langle B_C \phi_1, B_C \phi_2 \rangle_0 + \langle \dot{\phi}_1, \dot{\phi}_2 \rangle_0,$$

where

$$B_C := (-A + C_1C_2)^{1/2}.$$

Also for these generalized abstract wave equations we are able to construct singular perturbations \hat{W}_g of the skew-adjoint operator W_g which reduce to the previous ones in the case $C_1 = C_2 = 0$. Such singular perturbation, together with their resolvents, are defined in a relatively explicit way in terms of the original operators B , C_1 and C_2 .

The contents of the single sections are the following:

– *Section 2.* We review, with some variants and additions with respect to [7], [14] and [6] (and references therein) the theory of abstract wave equations. Here we are in particular interested (see Theorem 2.5) in computing the resolvent of W , the skew-adjoint operator corresponding to the abstract wave equation $\ddot{\phi} = -B^2\phi$, in terms of the resolvent of B^2 . For such a scope the scale of Hilbert spaces $\mathcal{H}_k := \{\phi \in \mathcal{H}_1 : \bar{B}\phi \in D(B^{k-1})\}$, $k \geq 1$, is used.

– *Section 3.* Given a continuous linear map $\tau : \bar{\mathcal{H}}_2 \rightarrow \mathfrak{h}$, \mathfrak{h} an auxiliary Hilbert space, such that, denoting by $\tau^* : \mathfrak{h} \rightarrow \mathcal{H}_{-2}$ the adjoint of the restriction of τ to $D(B^2)$, one has $\text{Ran}(\tau^*) \cap \mathcal{H}_{-1} = \emptyset$ (we are thus considering strongly singular perturbations of B^2), we construct, mimicking the approach developed in [11], skew-adjoint operators \hat{W} which coincide with W on $\text{Ker}(\tau) \oplus \mathcal{H}_1$. As already mentioned, due to our hypothesis on τ^* , the \hat{W} 's will be defined on a Hilbert space larger than $\bar{\mathcal{H}}_1 \oplus \mathcal{H}_0$, indeed it will be a space of the kind $\bar{\mathcal{H}}_1 \oplus \mathcal{H}_0 \oplus \mathfrak{h}$. Thus our strategy is the following: for any positive, bounded and injective self-adjoint operator Θ on \mathfrak{h} , at first we trivially extend W to $\bar{\mathcal{H}}_1 \oplus \mathcal{H}_0 \oplus \mathfrak{h}_\Theta$ (here \mathfrak{h}_Θ is the Hilbert space obtained from \mathfrak{h} by considering the scalar product induced by Θ) by defining $\tilde{W}(\phi, \dot{\phi}, \zeta) := (W(\phi, \dot{\phi}), 0)$, which is obviously still skew-adjoint. Then we consider the skew-symmetric operator obtained by restricting \tilde{W} to the kernel of the map τ_Θ , where $\tau_\Theta(\phi, \dot{\phi}, \zeta) := \tau\phi - \Theta\zeta$. To such a skew-symmetric operator, which depends on Θ , we apply the procedure given in [11], thus obtaining a family of skew-adjoint extensions parametrized by self-adjoint operators on \mathfrak{h} . Selecting from such a family the extension corresponding to the parametrizing operator zero, we obtain a skew-adjoint operator \tilde{W}_Θ which by construction coincides with \tilde{W} on the kernel of τ_Θ (see Theorem 3.4). Under the additional hypothesis that both the Hilbert spaces \mathcal{H}_0 and $\bar{\mathcal{H}}_1$ are contained in a common vector space (this is usually true in the case B is a (pseudo-)differential operator by considering some space of distributions), one can then define a suitable Hilbert space $\bar{\mathcal{K}}_1 \supset \bar{\mathcal{H}}_1$ and a skew-adjoint operator W_Θ on $\bar{\mathcal{K}}_1 \oplus \mathcal{H}_0$ such that W_Θ coincides with W on the set $\text{Ker}(\tau) \oplus \mathcal{H}_1$ (see Theorem 3.6). By our hypotheses such a set is dense in $\bar{\mathcal{H}}_1 \oplus \mathcal{H}_0$ and thus W_Θ is a singular perturbation of W .

The skew-adjoint operator W_Θ permits then to define $-A_\Theta$, an injective and positive self-adjoint operator on \mathcal{H}_0 which results to be a singular perturbation of $-A = B^2$. The resolvent and the quadratic form of A_Θ are also explicitly given. Regarding the quadratic form a variation on the Birman-Kreĭn-Vishik theory (see [3] and references therein) is obtained. Conversely, the skew-adjoint operator corresponding to the abstract wave equation $\ddot{\phi} = A_\Theta \phi$ results to be nothing but W_Θ (there results are summarized in Theorem 3.7). Thus we gave affirmative answers to questions 1-3 above.

– *Section 4.* We construct singular perturbations of the kind obtained in Section 3 for the skew-adjoint operator W_g corresponding now to the abstract wave equation $\ddot{\phi} - (C_1 + C_2)\dot{\phi} = -(B^2 + C_1C_2)\phi$. Here we put on the skew-adjoint operators C_1 and C_2 conditions which ensure that $B^2 + C_1C_2$ is self-adjoint, positive and injective. Defining $B_C := (B^2 + C_1C_2)^{1/2}$, $C := C_1 + C_2$, this case is studied by extending the procedure of Section 3 to the abstract wave equation $\ddot{\phi} - C\dot{\phi} = -B_C^2\phi$ (see Theorem 4.7). The analogues of Theorems 3.4 and 3.6 corresponding to the this more general situation are Theorems 4.8 and 4.11. Here an hypothesis concerning both C_1 , C_2 and a suitable extension $\bar{\tau}$ of the map τ must be introduced. Such hypothesis is surely verified when C_1 and C_2 are bounded operator, whereas its validity in the unbounded case is more subtle, as Example 3 in Section 5 shows.

– *Section 5.* We give some examples. In Example 1 we define skew-adjoint operators W_Θ , Θ an Hermitean injective and positive matrix on \mathbb{C}^n , corresponding to wave equations on star-like graphs with n open ends by defining singular perturbations of the skew-adjoint operator $W(\phi_1, \dots, \phi_n, \psi_1, \dots, \psi_n) := (\psi_1, \dots, \psi_n, \phi_1'', \dots, \phi_n'')$, where the ϕ 's are defined on the half-line $(0, \infty)$ and satisfy zero Dirichlet boundary conditions at the origin. The corresponding (according to Theorem 3.7) negative self-adjoint operator A_Θ is of the class of Laplacians on a star-like graphs (see [9] and references therein). By a similar construction, considering also second derivative operators on compact intervals, one could define wave equations on more complicated graphs.

In Example 2 we consider the case in which \mathcal{H}_0 is the space of square integrable functions on \mathbb{R}^3 , $B = (-\Delta)^{1/2}$, $C_1 = C_2 = 0$, $\mathfrak{h} = \mathbb{C}^n$ and $\tau\phi = (\phi(y_1), \dots, \phi(y_n))$, where $Y = \{y_1, \dots, y_n\}$ is a given discrete subset of \mathbb{R}^3 . This gives a singular perturbations of the free wave equations by n Dirac masses placed at points y_1, \dots, y_n , in the sense that the extensions constructed give a rigorous definition and provide existence of the dynamics for wave equations of the kind

$$\ddot{\phi} = \Delta\phi + \zeta_\phi^1 \delta_{y_1} + \dots + \zeta_\phi^n \delta_{y_n},$$

where $\zeta_\phi \equiv (\zeta_\phi^1, \dots, \zeta_\phi^n)$ is related to the value of the continuous part ϕ_0 of ϕ at the points in Y by the boundary conditions

$$\phi_0(y_i) = \sum_{1 \leq j \leq n} \theta_{ij} \zeta_\phi^j, \quad i = 1, \dots, n.$$

Such wave equations were introduced (by different methods) and analyzed, when $n = 1$, in [4]. The corresponding singular perturbation of the Laplacian, obtained according to Theorem 3.7 is of the class on point perturbation of the Laplacian (see [1] and references therein). The above situation can be generalized by taking as τ the evaluation map along a d -set (i.e. a d -dimensional Lipschitz submanifold if d is an integer or a self-similar fractal in the noninteger case), with and proceeding similarly to the examples appearing in [11]-[13], thus obtaining perturbations of the free wave equation supported on null sets. Here the extension parameter is a self-adjoint operator on some fractional order Sobolev space on the d -set.

A wave equation of the kind $\ddot{\phi} = \Delta\phi + 4\pi e M\zeta_\phi\delta_0$ was used to give a rigorous description of classical and quantum electrodynamics in dipole (or linear) approximation and without ultraviolet cut-off (see [10] and [5]). Here ϕ is \mathbb{R}^3 -valued and plays the role of the electromagnetic potential in the Coulomb gauge (thus $\text{div } \phi = 0$), M is the projector onto the divergenceless fields and e is the electric charge (the velocity of light being set to be equal to one). In this case one must modify the above boundary condition (here $Y = \{0\}$), considering the (no more linear but affine) one given by

$$\phi_0(0) = -\frac{m}{e} \zeta_\phi + \frac{1}{e} p,$$

where p is an arbitrary vector in \mathbb{R}^3 and m is the mass of the particle. In this framework $\zeta_\phi \in \mathbb{R}^3$ can be identified with the particle velocity v , so that the particle dynamics is given by the evolution of the field singularity. With this identification the above boundary condition is nothing else than the usual (linearized and regularized) relation between velocity and momentum (represented by the vector p) in the presence of an electromagnetic field, i.e. $p = mv + e\phi_0(0)$.

This approach suggests that the study of singular perturbations of the wave equation $\ddot{\phi} = \Delta\phi$ can produce a useful framework for a rigorous treatment of classical electrodynamics of point particles and for quantum electrodynamics in the ultraviolet limit. Indeed this was the original motivation of the paper. In order to remove the limitation given by the dipole approximation assumed in [10] and [5], one is led to study the singular perturbations, supported at the origin, of the

wave equation

$$\begin{aligned}\dot{\phi} &= v \cdot \nabla \phi + \psi \\ \dot{\psi} &= v \cdot \nabla \psi + \Delta \phi,\end{aligned}$$

where v is a given vector in \mathbb{R}^3 with $|v| < 1$. This is suggested by starting with the Maxwell-Lorentz system, by re-writing it in a reference frame co-moving with the particle and then by performing the reduction allowed by the conservation of the total (particle + field) momentum. We refer to the digression given at the end of Section 5 for a more detailed discussion. Thus in the successive example in Section 5 (Example 3), we modify the situation considered in Example 2 (in the case $Y = \{0\}$) by taking $C_1 = C_2 = v \cdot \nabla$, with $v \in \mathbb{R}^3$, $|v| < 1$. In this case the regular part ϕ_0 of ϕ 's in the proper operator domain is no more continuous (when $v \neq 0$) and the evaluation map of Example 2 has to be extended to $\bar{\tau}$, where $\bar{\tau}\phi_0$ is defined by the limit $R \downarrow 0$ of the average $\langle \phi_0 \rangle_R$ of ϕ_0 over the sphere of radius R . It is here proven that such a limit exists for the functions in the operator domain of the extensions. This produces a rigorous definition and existence of the dynamics for the wave equation

$$\begin{aligned}\dot{\phi} &= v \cdot \nabla \phi + \psi \\ \dot{\psi} &= v \cdot \nabla \psi + \Delta \phi + \zeta_\phi \delta_0,\end{aligned}$$

where now the ζ_ϕ 's are related to the regular part ϕ_0 of the ϕ 's by the boundary condition

$$\langle \phi_0 \rangle := \lim_{R \downarrow 0} \langle \phi_0 \rangle_R = \theta \zeta_\phi.$$

Once the proper domain of definition for the fields ϕ and ψ is determined by this linear analysis, a nonlinear operator, candidate to describe the classical electrodynamics of a point particle, can be obtained by considering the nonlinear wave equation

$$\begin{aligned}\dot{\phi} &= v \cdot \nabla \phi + \psi \\ \dot{\psi} &= v \cdot \nabla \psi + \Delta \phi + 4\pi e M v \delta_0,\end{aligned}$$

where v , again representing the particle velocity, is no more a given vector but is related to the regular parts ϕ_0 and ψ_0 of the fields ϕ and

ψ by the nonlinear boundary condition

$$\langle \phi_0 \rangle - \frac{1}{4\pi e} \langle \psi_0, \nabla \phi_0 \rangle = -\frac{m}{e} \frac{v}{\sqrt{1-|v|^2}} + \frac{1}{e} \Pi.$$

The (conserved) total momentum Π of the particle-field system is defined, in terms of the particle momentum p , by $\Pi := p - \frac{1}{4\pi} \langle \psi, \nabla \phi \rangle$. Thus the above boundary condition corresponds to the (regularized) velocity-momentum relation for a (relativistic) particle in the presence of an electromagnetic field, i.e. $p = \frac{mv}{\sqrt{1-|v|^2}} + e \langle \phi_0 \rangle$. Again we refer to the digression at the end of Section 5 for more details.

– *Appendix.* We give a compact review of the approach to singular perturbations of self-adjoint operators developed in [11] adapted to our present (skew-adjoint) situation. In particular, with reference to the notations in [11], we make here a particular choice of the operator Γ which correspond, in the case treated in Section 3 here, to a weakly singular perturbation. Thus a strongly singular perturbation \hat{A} of A gives rise to a weakly singular perturbation \hat{W} of W . This could be used to study the scattering theory for strongly singular perturbations of A in terms of weakly singular perturbations. Indeed, by Birman-Kato invariance principle, the Möller operators $\Omega_{\pm}(\hat{W}, W)$ and $\Omega_{\pm}(\hat{A}, A)$ are unitarily equivalent. As regard the parametrizing operator, as we already said above, we pick up here, in the family of skew-adjoint extensions given by the general scheme in [11], the extension corresponding to the zero operator.

2. ABSTRACT WAVE EQUATIONS

Let $B : D(B) \subseteq \mathcal{H}_0 \rightarrow \mathcal{H}_0$ be a self-adjoint operator on the Hilbert space \mathcal{H}_0 such that $\text{Ker}(B) = \{0\}$. Let us denote by \mathcal{H}_k , $k > 0$, the scale of Hilbert spaces given by the domain of B^k with the scalar product $\langle \cdot, \cdot \rangle_k$ leading to the graph norm, i.e.

$$\langle \phi_1, \phi_2 \rangle_k := \langle B^k \phi_1, B^k \phi_2 \rangle_0 + \langle \phi_1, \phi_2 \rangle_0.$$

Here $\langle \cdot, \cdot \rangle_0$ denotes the scalar product in \mathcal{H}_0 . We will use the symbol $\| \cdot \|_0$ to indicate the corresponding norm.

We then define the Hilbert space $\bar{\mathcal{H}}_1$ by completing the pre-Hilbert space $D(B)$ endowed with the scalar product

$$[\phi_1, \phi_2]_1 := \langle B\phi_1, B\phi_2 \rangle_0.$$

We define $\bar{B} \in \mathcal{B}(\bar{\mathcal{H}}_1, \mathcal{H}_0)$ as the closed bounded extension of the densely defined linear operator

$$B : \mathcal{H}_1 \subseteq \bar{\mathcal{H}}_1 \rightarrow \mathcal{H}.$$

Here and below by $\mathbf{B}(X, Y)$ we mean the space of bounded, everywhere defined, linear operators on the Banach space X to the Banach space Y ; for brevity we put $\mathbf{B}(X) \equiv \mathbf{B}(X, X)$.

Since B is self-adjoint one has

$$\text{Ran}(B)^\perp = \text{Ker}(B),$$

so that, B being injective, $\text{Ran}(B)$ is dense in \mathcal{H}_0 . Therefore we can define $\bar{B}^{-1} \in \mathbf{B}(\mathcal{H}_0, \bar{\mathcal{H}}_1)$ as the closed bounded extension of the densely defined linear operator

$$B^{-1} : \text{Ran}(B) \subseteq \mathcal{H}_0 \rightarrow \bar{\mathcal{H}}_1.$$

One can then verify that \bar{B} is boundedly invertible with inverse given by \bar{B}^{-1} .

Given \bar{B} we introduce the scale of spaces $\bar{\mathcal{H}}_k$, $k \geq 1$, defined by

$$\bar{\mathcal{H}}_k := \{ \phi \in \bar{\mathcal{H}}_1 : \bar{B}\phi \in \mathcal{H}_{k-1} \}.$$

Obviously $\mathcal{H}_k \subseteq \bar{\mathcal{H}}_k$.

Lemma 2.1.

$$\bar{\mathcal{H}}_k = \mathcal{H}_k + \bar{\mathcal{H}}_{k+1}.$$

Proof. The thesis follow from

$$\bar{\mathcal{H}}_{2k} = \bar{B}^{-1}(B + i)^{-1}(B^{2(k-1)} + 1)^{-1}(\mathcal{H}_0),$$

$$\mathcal{H}_{2k} = (B^2 + 1)^{-1}(B^{2(k-1)} + 1)^{-1}(\mathcal{H}_0),$$

$$\bar{\mathcal{H}}_{2k+1} = \bar{B}^{-1}(B^{2k} + 1)^{-1}(\mathcal{H}_0),$$

$$\mathcal{H}_{2k+1} = (B + i)^{-1}(B^{2k} + 1)^{-1}(\mathcal{H}_0),$$

and from the identities

$$\bar{B}^{-1} = (B + i)^{-1} + i\bar{B}^{-1}(B + i)^{-1},$$

$$\bar{B}^{-1}(B + i)^{-1} = (B^2 + 1)^{-1} - i\bar{B}^{-1}(B^2 + 1)^{-1}.$$

□

Lemma 2.2. *The set $\bar{\mathcal{H}}_k$ endowed with the scalar product*

$$[\phi_1, \phi_2]_k := \langle \bar{B}\phi_1, \bar{B}\phi_2 \rangle_{k-1}$$

is a Hilbert space.

Proof. Let ϕ_n , $n \geq 1$, be a Cauchy sequence in $\bar{\mathcal{H}}_k$. Then ϕ_n , $n \geq 1$, is Cauchy in $\bar{\mathcal{H}}_1$ and $\bar{B}\phi_n$, $n \geq 1$, is Cauchy in $\bar{\mathcal{H}}^{k-1}$. Thus $\bar{B}\phi_n \rightarrow \bar{B}\phi$ and $B^{k-1}\bar{B}\phi_n \rightarrow \psi$ in \mathcal{H}_0 . Since B^{k-1} is closed, $\bar{B}\phi \in \mathcal{H}_{k-1}$, hence $\phi \in \bar{\mathcal{H}}_k$, and $\psi = B^{k-1}\bar{B}\phi$. □

Remark 2.3. The previous lemma shows that $\bar{\mathcal{H}}_k$ could be alternatively defined as the completion of pre-Hilbert space $D(B^k)$ endowed with the scalar product

$$[\phi_1, \phi_2]_k := \langle B\phi_1, B\phi_2 \rangle_{k-1}.$$

Thus \mathcal{H}_k is dense in $\bar{\mathcal{H}}_k$.

We now define

$$\bar{A} : \bar{\mathcal{H}}_2 \rightarrow \mathcal{H}_0, \quad \bar{A} := -B\bar{B}.$$

Remark 2.4. By the previous remark $\bar{A} \in \mathcal{B}(\bar{\mathcal{H}}_2, \mathcal{H}_0)$ could be alternatively defined as the closed bounded extension of the densely defined linear operator $A := -B^2 : \mathcal{H}_2 \subseteq \bar{\mathcal{H}}_2 \rightarrow \mathcal{H}_0$.

We put, for any real $\lambda \neq 0$,

$$R_0(\lambda) := (B^2 + \lambda^2)^{-1}, \quad R_0(\lambda) \in \mathcal{B}(\mathcal{H}_0, \mathcal{H}_2)$$

and then define $\bar{R}_0(\lambda) \in \mathcal{B}(\bar{\mathcal{H}}_1, \bar{\mathcal{H}}_3)$ as the closed bounded extension of

$$R_0(\lambda) : \mathcal{H}_1 \subseteq \bar{\mathcal{H}}_1 \rightarrow \bar{\mathcal{H}}_3.$$

The linear operator $\bar{R}_0(\lambda)$ satisfies the relations

$$(2.1) \quad -\bar{A}\bar{R}_0(\lambda) + \lambda^2\bar{R}_0(\lambda) = \mathbf{1}_{\bar{\mathcal{H}}_1},$$

$$(2.2) \quad -R_0(\lambda)\bar{A} + \lambda^2\bar{R}_0(\lambda) = \mathbf{1}_{\bar{\mathcal{H}}_2},$$

On the Hilbert space $\bar{\mathcal{H}}_1 \oplus \mathcal{H}_0$ with scalar product given by

$$\langle\langle (\phi_1, \psi_1), (\phi_2, \psi_2) \rangle\rangle := \langle \bar{B}\phi_1, \bar{B}\phi_2 \rangle_0 + \langle \psi_1, \psi_2 \rangle_0.$$

we define the linear operator

$$W : \bar{\mathcal{H}}_2 \oplus \mathcal{H}_1 \subseteq \bar{\mathcal{H}}_1 \oplus \mathcal{H}_0 \rightarrow \bar{\mathcal{H}}_1 \oplus \mathcal{H}_0, \quad W(\phi, \psi) := (\psi, \bar{A}\phi).$$

Theorem 2.5. *The linear operator W is skew-adjoint and its resolvent is given by*

$$(-W + \lambda)^{-1}(\phi, \psi) = (\lambda\bar{R}_0(\lambda)\phi + R_0(\lambda)\psi, -\phi + \lambda^2\bar{R}_0(\lambda)\phi + \lambda R_0(\lambda)\psi).$$

Proof. The skew-symmetry of W immediately follows from the definition of the scalar product $\langle\langle \cdot, \cdot \rangle\rangle$. The fact that $(-W + \lambda)^{-1}$ as defined above is the inverse of $-W + \lambda$ is a matter of algebraic computations given the definition of $R_0(\lambda)$, $\bar{R}_0(\lambda)$ and (2.1), (2.2). The proof is then concluded by recalling that W is skew-adjoint (equivalently iW is self-adjoint) if and only if it is skew-symmetric and $\text{Ran}(W \pm \lambda) = \bar{\mathcal{H}}_1 \oplus \mathcal{H}_0$ for some real $\lambda \neq 0$. \square

Remark 2.6. Note that $\bar{\mathcal{H}}_2 \oplus \mathcal{H}_1 = \text{Ran}(-W + \lambda)^{-1}$ gives a decomposition compatible with the one given by lemma 1.1, i.e. $\bar{\mathcal{H}}_2 = \mathcal{H}_2 + \bar{\mathcal{H}}_3$ and $\mathcal{H}_1 = \bar{\mathcal{H}}_1 + \mathcal{H}_2 + \bar{\mathcal{H}}_3$.

Remark 2.7. Note that the norm on $\bar{\mathcal{H}}_2$ induced by the graph norm of W coincides with the one given by the scalar product $[\cdot, \cdot]_2$. Hence the domain of W is the direct sum of the Hilbert spaces $\bar{\mathcal{H}}_2$ and \mathcal{H}_1 as written above.

3. SINGULAR PERTURBATIONS OF ABSTRACT WAVE EQUATIONS

On the Hilbert space \mathfrak{h} with scalar product $\langle \cdot, \cdot \rangle_{\mathfrak{h}}$ and norm $\|\cdot\|_{\mathfrak{h}}$, we consider a bounded, positive and injective self-adjoint operator Θ . Then we denote by \mathfrak{h}_{Θ} the Hilbert space given by \mathfrak{h} endowed with the scalar product

$$\langle \zeta_1, \zeta_2 \rangle_{\Theta} := \langle \Theta \zeta_1, \zeta_2 \rangle_{\mathfrak{h}}.$$

The corresponding norm will be indicated by $\|\cdot\|_{\Theta}$.

By Theorem 2.5, on Hilbert space $\bar{\mathcal{H}}_1 \oplus \mathcal{H}_0 \oplus \mathfrak{h}_{\Theta}$ with scalar product

$$\langle (\phi_1, \psi_1, \zeta_1), (\phi_2, \psi_2, \zeta_2) \rangle_{\Theta} := \langle \bar{B}\phi_1, \bar{B}\phi_2 \rangle_0 + \langle \psi_1, \psi_2 \rangle_0 + \langle \zeta_1, \zeta_2 \rangle_{\Theta},$$

the linear operator

$$\begin{aligned} \tilde{W} : \bar{\mathcal{H}}_2 \oplus \mathcal{H}_1 \oplus \mathfrak{h}_{\Theta} &\subseteq \bar{\mathcal{H}}_1 \oplus \mathcal{H}_0 \oplus \mathfrak{h}_{\Theta} \rightarrow \bar{\mathcal{H}}_1 \oplus \mathcal{H}_0 \oplus \mathfrak{h}_{\Theta}, \\ \tilde{W}(\phi, \psi, \zeta) &:= (W(\psi, \phi), 0) \end{aligned}$$

is skew-adjoint and

$$(3.1) \quad (-\tilde{W} + \lambda)^{-1}(\phi, \psi, \zeta) = ((-W + \lambda)^{-1}(\phi, \psi), \lambda^{-1}\zeta).$$

Given $\tau \in \mathcal{B}(\bar{\mathcal{H}}_2, \mathfrak{h})$, we define $\tau_{\Theta} \in \mathcal{B}(\bar{\mathcal{H}}_2 \oplus \mathcal{H}_0 \oplus \mathfrak{h}_{\Theta}, \mathfrak{h})$ by

$$\tau_{\Theta} : \bar{\mathcal{H}}_2 \oplus \mathcal{H}_0 \oplus \mathfrak{h}_{\Theta} \rightarrow \mathfrak{h}, \quad \tau_{\Theta}(\phi, \psi, \zeta) := \tau\phi - \Theta\zeta.$$

The action of τ_{Θ} satisfies A.1 (see the appendix). Now we suppose that it also satisfies A.2, i.e. we suppose H3.0)

$$\text{Ran}(\tau_{\Theta}) = \mathfrak{h}.$$

Of course H3.0 holds true if τ itself is surjective. Another possibility is

$$\forall \zeta \in \mathfrak{h}, \quad \|\Theta\zeta\|_{\mathfrak{h}} \geq c \|\zeta\|_{\mathfrak{h}}, \quad c > 0,$$

which is equivalent to $\text{Ran}(\Theta) = \mathfrak{h}$.

Now we define $\check{G}(\lambda) \in \mathcal{B}(\mathcal{H}_0, \mathfrak{h})$ and $G(\lambda) \in \mathcal{B}(\mathfrak{h}, \mathcal{H}_0)$ by

$$\check{G}(\lambda) := \tau R_0(\lambda), \quad G(\lambda) := \check{G}(\lambda)^*.$$

We also define $\check{\bar{G}}(\lambda) \in \mathcal{B}(\bar{\mathcal{H}}_1, \mathfrak{h})$ and $\bar{G}(\lambda) \in \mathcal{B}(\mathfrak{h}, \bar{\mathcal{H}}_1)$ by

$$\check{\bar{G}}(\lambda) := \tau \bar{R}_0(\lambda), \quad \bar{G}(\lambda) := \check{\bar{G}}(\lambda)^*.$$

Obviously $\check{G}(\lambda) = \check{G}(\lambda)$ on \mathcal{H}_1 .

Lemma 3.1.

$$\text{Ran}(G(\lambda)) \subseteq \text{Ran}(B) \quad \text{and} \quad \bar{G}(\lambda) = \bar{B}^{-1}B^{-1}G(\lambda) \in \mathbf{B}(\mathfrak{h}, \bar{\mathcal{H}}_2).$$

Proof. By the definitions of $\bar{G}(\lambda)$ and $G(\lambda)$ one has, for any $\zeta \in \mathfrak{h}$ and for any $\psi \in \mathcal{H}_1$,

$$\langle B\psi, \bar{B}\bar{G}(\lambda)\zeta \rangle_0 = \langle \check{G}(\lambda)\psi, \zeta \rangle_{\mathfrak{h}} = \langle \psi, G(\lambda)\zeta \rangle_0.$$

Being B self-adjoint with domain \mathcal{H}_1 , the above relation shows that $\bar{B}\bar{G}(\lambda)\zeta \in \mathcal{H}_1$, hence $\bar{G}(\lambda)\zeta \in \bar{\mathcal{H}}_2$,

$$B\bar{B}\bar{G}(\lambda) = G(\lambda),$$

$$\|\bar{B}\bar{G}(\lambda)\zeta\|_0^2 + \|B\bar{B}\bar{G}(\lambda)\zeta\|_0^2 = \|\bar{B}\bar{G}(\lambda)\zeta\|_0^2 + \|G(\lambda)\zeta\|_0^2.$$

□

Defining $\check{G}_{\Theta}(\lambda) \in \mathbf{B}(\bar{\mathcal{H}}_1 \oplus \mathcal{H}_0 \oplus \mathfrak{h}_{\Theta}, \mathfrak{h})$ by

$$(3.2) \quad \check{G}_{\Theta}(\lambda)(\phi, \psi, \zeta) := \tau_{\Theta}(-\tilde{W} + \lambda)^{-1}(\phi, \psi, \zeta) = \lambda\check{G}(\lambda)\phi + \check{G}(\lambda)\psi - \lambda^{-1}\Theta\zeta$$

and $G_{\Theta}(\lambda) \in \mathbf{B}(\mathfrak{h}, \bar{\mathcal{H}}_1 \oplus \mathcal{H}_0 \oplus \mathfrak{h}_{\Theta})$ by

$$(3.3) \quad G_{\Theta}(\lambda)\zeta := -\check{G}_{\Theta}(-\lambda)^*\zeta = (\lambda\bar{G}(\lambda)\zeta, -G(\lambda)\zeta, -\lambda^{-1}\zeta),$$

one has that, by the previous lemma, $G_{\Theta}(\lambda) \in \mathbf{B}(\mathfrak{h}, \bar{\mathcal{H}}_2 \oplus \mathcal{H}_0 \oplus \mathfrak{h}_{\Theta})$. Thus A.4 is satisfied,

$$(3.4) \quad \Gamma_{\Theta}(\lambda) := -\tau_{\Theta}G_{\Theta}(\lambda) = -\lambda\tau\bar{G}(\lambda) - \frac{1}{\lambda}\Theta$$

$$(3.5) \quad = -\lambda\tau\bar{B}^{-1}B^{-1}G(\lambda) - \frac{1}{\lambda}\Theta$$

is well-defined and $\Gamma_{\Theta}(\lambda) \in \mathbf{B}(\mathfrak{h})$. Let us now show that A.5 is satisfied:

Lemma 3.2.

$$\Gamma_{\Theta}(\lambda)^* = -\Gamma_{\Theta}(-\lambda).$$

Proof. By [11], Lemma 2.1,

$$(\lambda^2 - \epsilon^2)R_0(\epsilon)G(\lambda) = G(\epsilon) - G(\lambda).$$

Since $\text{Ran}(G(\lambda)) \subseteq \text{Ran}(B)$, $R_0(\epsilon)G(\lambda)$ strongly converges in $\mathbf{B}(\mathfrak{h}, \bar{\mathcal{H}}_2)$, as $\epsilon \downarrow 0$, to $\bar{B}^{-1}B^{-1}G(\lambda)$ when $B^2R_0(\epsilon)$ strongly converges to the

identity operator on \mathcal{H}_0 . Since B^2 is injective this follows proceeding as in [12], Section 3. Therefore one has that

$$\begin{aligned}\Gamma_\Theta(\lambda) &= \text{s-}\lim_{\epsilon \downarrow 0} -\frac{1}{\lambda} (\Theta + \tau(G(\epsilon) - G(\lambda))) \\ &= \text{s-}\lim_{\epsilon \downarrow 0} -\frac{1}{\lambda} \left(\Theta + (\lambda^2 - \epsilon^2) \check{G}(\epsilon) G(\lambda) \right) \\ &= \text{s-}\lim_{\epsilon \downarrow 0} -\frac{1}{\lambda} \left(\Theta + (\lambda^2 - \epsilon^2) \check{G}(\lambda) G(\epsilon) \right) .\end{aligned}$$

The proof is concluded by observing that $\tau(G(\epsilon)) - G(\lambda)$ is symmetric (see [11], Lemma 2.2. Also see [12], Lemma 3). \square

Remark 3.3. By the same methods used in the above proof (i.e. using the fact that $\text{Ran}(G(\lambda)) \subseteq \text{Ran}(B)$), all the results contained in [12] can be extended to the case in which $\tau \in \mathbf{B}(\mathcal{H}_2, \mathfrak{h})$, thus allowing for the treatment of singular perturbations of convolution operators also in lower dimensions (in [12] the examples were given in \mathbb{R}^d with $d \geq 4$).

Denote by \mathcal{H}_{-k} , $k \geq 0$, the completion of \mathcal{H}_0 with respect to the scalar product

$$\langle \phi_1, \phi_2 \rangle_{-k} := \langle (B^{2k} + 1)^{-1/2} \phi_1, (B^{2k} + 1)^{-1/2} \phi_2 \rangle_0 .$$

Of course $\mathcal{H}_{-k} \subseteq \mathcal{H}_{-(k+1)}$. Since $\tau \in \mathbf{B}(\mathcal{H}_2, \mathfrak{h})$ we define $\tau^* \in \mathbf{B}(\mathfrak{h}, \mathcal{H}_{-2})$ by

$$\langle (B^4 + 1)^{-1/2} \tau^* \zeta, (B^4 + 1)^{1/2} \phi \rangle_0 = \langle \zeta, \tau \phi \rangle_{\mathfrak{h}}, \quad \zeta \in \mathfrak{h}, \phi \in \mathcal{H}_2 .$$

Now we suppose that

H3.1)

$$\text{Ran}(\tau^*) \cap \mathcal{H}_{-1} = \{0\} .$$

This, using the definition of $G(\lambda)$, is equivalent to

$$\text{Ran}(G(\lambda)) \cap \mathcal{H}_1 = \{0\} ,$$

so that A.3 is satisfied, i.e.

$$\text{Ran}(G_\Theta(\lambda)) \cap D(\tilde{W}) = \{0\} .$$

By Theorem 6.2 we can define a skew-adjoint extension of the skew-symmetric operator given by the restriction of \tilde{W} to the dense set

$$\mathcal{N}_\Theta := \{(\phi, \psi, \zeta) \in \bar{\mathcal{H}}_2 \oplus \mathcal{H}_1 \oplus \mathfrak{h}_\Theta : \tau \phi = \Theta \zeta\} :$$

Theorem 3.4. *Suppose that H3.0 and H3.1 hold true. Let*

$$\begin{aligned}D(\tilde{W}_\Theta) &:= \{(\phi_0, \psi, \zeta_\phi) \in \bar{\mathcal{H}}_1 \oplus \mathcal{H}_0 \oplus \mathfrak{h}_\Theta : \phi_0 \in \bar{\mathcal{H}}_2, \\ &\psi = \psi_\lambda + G(\lambda) \zeta_\psi, \quad \psi_\lambda \in \mathcal{H}_1, \zeta_\psi \in \mathfrak{h}, \quad \Theta \zeta_\phi = \tau \phi_0\} .\end{aligned}$$

Then

$$\tilde{W}_\Theta : D(\tilde{W}_\Theta) \subseteq \bar{\mathcal{H}}_1 \oplus \mathcal{H}_0 \oplus \mathfrak{h}_\Theta \rightarrow \bar{\mathcal{H}}_1 \oplus \mathcal{H}_0 \oplus \mathfrak{h}_\Theta ,$$

$$\tilde{W}_\Theta(\phi_0, \psi, \zeta_\phi) := (\psi_0, \bar{A}\phi_0, \zeta_\psi) ,$$

is a skew-adjoint extension of the restriction of \tilde{W} to the dense set \mathcal{N}_Θ . Here $\psi_0 \in \bar{\mathcal{H}}_1$, defined by

$$\psi_0 := \psi_\lambda - \lambda^2 \bar{B}^{-1} B^{-1} G(\lambda) \zeta_\psi ,$$

does not depend on λ . The resolvent of \tilde{W}_Θ is given by

$$(-\tilde{W}_\Theta + \lambda)^{-1} = (-\tilde{W} + \lambda)^{-1} + G_\Theta(\lambda) \Gamma_\Theta(\lambda)^{-1} \check{G}_\Theta(\lambda) ,$$

where the bounded linear operators $(-\tilde{W} + \lambda)^{-1}$, $\check{G}_\Theta(\lambda)$, $G_\Theta(\lambda)$, $\Gamma_\Theta(\lambda)^{-1}$ have been defined in (3.1)-(3.4) respectively.

Proof. By Theorem 6.4 we know that $(-\tilde{W} + \lambda)^{-1} + G_\Theta(\lambda) \Gamma_\Theta(\lambda)^{-1} \check{G}_\Theta(\lambda)$ is the resolvent of a skew-adjoint extension \hat{W}_Θ of the restriction of \tilde{W} to the dense set \mathcal{N}_Θ . Therefore $(\hat{\phi}_0, \hat{\psi}, \hat{\zeta}_\phi) \in D(\hat{W}_\Theta)$ if and only if

$$\begin{aligned} \hat{\phi}_0 &= \phi_\lambda + \lambda \bar{G}(\lambda) \Gamma_\Theta(\lambda)^{-1} (\tau \phi_\lambda - \Theta \zeta_\lambda) , & \phi_\lambda &\in \bar{\mathcal{H}}_2 , \\ \hat{\psi} &= \psi_\lambda - G(\lambda) \Gamma_\Theta(\lambda)^{-1} (\tau \phi_\lambda - \Theta \zeta_\lambda) , & \psi_\lambda &\in \mathcal{H}_1 , \\ \hat{\zeta}_\phi &= \zeta_\lambda - \frac{1}{\lambda} \Gamma_\Theta(\lambda)^{-1} (\tau \phi_\lambda - \Theta \zeta_\lambda) , & \zeta_\lambda &\in \mathfrak{h} . \end{aligned}$$

Let us now show that $D(\hat{W}_\Theta) = D(\tilde{W}_\Theta)$.

Since $\text{Ran}(\bar{G}(\lambda)) \subseteq \bar{\mathcal{H}}_2$, so that $\hat{\phi}_0 \in \bar{\mathcal{H}}_2$, and

$$\begin{aligned} &\tau_\Theta((-\tilde{W} + \lambda)^{-1} + G_\Theta(\lambda) \Gamma_\Theta(\lambda)^{-1} \check{G}_\Theta(\lambda)) \\ &= \check{G}_\Theta(\lambda) - \Gamma_\Theta(\lambda) \Gamma_\Theta(\lambda)^{-1} \check{G}_\Theta(\lambda) = 0 , \end{aligned}$$

so that $\tau \hat{\phi}_0 = \Theta \hat{\zeta}_\phi$, we have $D(\hat{W}_\Theta) \subseteq D(\tilde{W}_\Theta)$. Let us now prove the reverse inclusion. Given $(\phi_0, \psi, \zeta_\phi) \in D(\tilde{W}_\Theta)$ let us define

$$\begin{aligned} \phi_\lambda &:= \phi_0 + \lambda \bar{G}(\lambda) \zeta_\psi , \\ \zeta_\lambda &:= \zeta - \frac{1}{\lambda} \zeta_\psi . \end{aligned}$$

Then

$$\tau \phi_0 = \tau \phi_\lambda - \lambda \tau \bar{G}(\lambda) \zeta_\psi = \Theta \zeta = \Theta \left(\zeta_\lambda + \frac{1}{\lambda} \zeta_\psi \right)$$

implies

$$\tau \phi_\lambda - \Theta \zeta_\lambda = \left(\lambda \tau \bar{G}(\lambda) + \frac{1}{\lambda} \Theta \right) \zeta_\psi ,$$

i.e.

$$\zeta_\psi = -\Gamma_\Theta(\lambda)^{-1} (\tau \phi_\lambda - \Theta \zeta_\lambda) .$$

Thus $D(\tilde{W}_\Theta) \subseteq D(\hat{W}_\Theta)$. Now we have

$$\begin{aligned} \hat{W}_\Theta(\phi_0, \psi, \zeta_\phi) &= \tilde{W}(\phi_\lambda, \psi_\lambda, \zeta_\lambda) + \lambda(\phi_0 - \phi_\lambda, \psi - \psi_\lambda, \zeta_\phi - \zeta_\lambda) \\ &= (\psi_\lambda - \lambda^2 \bar{G}(\lambda) \zeta_\psi, \bar{A}\phi_\lambda + \lambda G(\lambda) \zeta_\psi, \zeta_\psi) \\ &= (\psi_0, \bar{A}(\phi_\lambda - \lambda \bar{G}(\lambda) \zeta_\psi), \zeta_\psi) \\ &= (\psi_0, \bar{A}\phi_0, \zeta_\psi) \\ &= \tilde{W}_\Theta(\phi_0, \psi, \zeta_\phi). \end{aligned}$$

ψ_0 does not depend on λ since the definition of \tilde{W}_Θ is λ -independent. \square

Let us now suppose that

H3.2) both \mathcal{H}_0 and $\bar{\mathcal{H}}_1$ are contained in a given vector space \mathcal{V} .

Thus we can define

$$G : \mathfrak{h} \rightarrow \mathcal{V}, \quad G := G(\lambda) + \lambda^2 \bar{G}(\lambda).$$

Lemma 3.5. *The definition of G is λ -independent. Moreover*

$$\text{Ran}(G) \cap \bar{\mathcal{H}}_1 = \{0\}.$$

Proof. By first resolvent identity one has (see [11], Lemma 2.1)

$$(\lambda^2 - \mu^2) R_0(\mu) G(\lambda) = G(\mu) - G(\lambda),$$

i.e.

$$\lambda^2 G(\lambda) - \mu^2 G(\mu) = B^2(G(\mu) - G(\lambda)).$$

This implies, by Lemma 3.1,

$$G(\lambda) + \lambda^2 \bar{G}(\lambda) = G(\mu) + \mu^2 \bar{G}(\mu).$$

Suppose there exists $\zeta \in \mathfrak{h}$ such that

$$G(\lambda)\zeta + \lambda^2 \bar{G}(\lambda)\zeta = \phi \in \bar{\mathcal{H}}_1.$$

Then $G(\lambda)\zeta \in \mathcal{H}_1$ and so, by H3.1, $G(\lambda)\zeta = 0$. By Lemma 3.1 $\bar{G}(\lambda)\zeta = 0$ and the proof is done. \square

By the previous lemma the following spaces are well-defined:

$$\bar{\mathcal{K}}_1 := \{\phi \in \mathcal{V} : \phi = \phi_0 + G\zeta_\phi, \phi_0 \in \bar{\mathcal{H}}_1, \zeta_\phi \in \mathfrak{h}\},$$

$$\bar{\mathcal{K}}_2 := \{\phi \in \mathcal{V} : \phi = \phi_0 + G\zeta_\phi, \phi_0 \in \bar{\mathcal{H}}_2, \zeta_\phi \in \mathfrak{h}\},$$

$$\mathcal{K}_1 := \bar{\mathcal{K}}_1 \cap \mathcal{H}_0.$$

Moreover the map

$$U : \bar{\mathcal{H}}_1 \oplus \mathcal{H}_0 \oplus \mathfrak{h}_\Theta \rightarrow \bar{\mathcal{K}}_1 \oplus \mathcal{H}_0, \quad U(\phi_0, \psi, \zeta_\phi) := (\phi_0 + G\zeta_\phi, \psi)$$

is injective and surjective and thus is unitary once we make $\bar{\mathcal{K}}_1$ a Hilbert space by defining the scalar product

$$\langle \phi, \varphi \rangle_{\bar{\mathcal{K}}_1} := \langle \bar{B}\phi_0, \bar{B}\varphi_0 \rangle_0 + \langle \zeta_\phi, \zeta_\varphi \rangle_\Theta.$$

Thus we can state the following:

Theorem 3.6. *Suppose that H3.0, H3.1 and H3.2 hold true. Then the linear operator*

$$\begin{aligned} W_\Theta : D(W_\Theta) &\subseteq \bar{\mathcal{K}}_1 \oplus \mathcal{H}_0 \rightarrow \bar{\mathcal{K}}_1 \oplus \mathcal{H}_0, \\ D(W_\Theta) &= \{(\phi, \psi) \in \bar{\mathcal{K}}_2 \oplus \mathcal{K}_1 : \Theta \zeta_\phi = \tau \phi_0\}, \\ W_\Theta(\phi, \psi) &:= U\tilde{W}_\Theta U^*(\phi, \psi) = (\psi, \bar{A}\phi_0). \end{aligned}$$

is skew-adjoint. It coincides with

$$W : \bar{\mathcal{H}}_2 \oplus \mathcal{H}_1 \subseteq \bar{\mathcal{H}}_1 \oplus \mathcal{H}_0 \rightarrow \bar{\mathcal{H}}_1 \oplus \mathcal{H}_0, \quad W(\phi, \psi) = (\psi, \bar{A}\phi)$$

on the dense set

$$D(W) \cap D(W_\Theta) = \{\phi \in \bar{\mathcal{H}}_2 : \tau \phi = 0\} \oplus \mathcal{H}_1.$$

Once we obtained W_Θ we can define the linear operator A_Θ on \mathcal{H}_0 by

$$\begin{aligned} D(A_\Theta) &:= \{\phi \in \bar{\mathcal{K}}_2 \cap \mathcal{H}_0 : \Theta \zeta_\phi = \tau \phi_0\}, \\ A_\Theta : D(A_\Theta) &\subseteq \mathcal{H}_0 \rightarrow \mathcal{H}_0, \quad A_\Theta \phi := P_2 W_\Theta I_1 \phi \equiv \bar{A}\phi_0, \end{aligned}$$

where

$$P_2 : \bar{\mathcal{K}}_1 \oplus \mathcal{H}_0 \rightarrow \mathcal{H}_0 \quad P_2(\phi, \psi) := \psi,$$

and

$$I_1 : \bar{\mathcal{K}}_2 \cap \mathcal{H}_0 \rightarrow \bar{\mathcal{K}}_2 \oplus \mathcal{K}_1, \quad I_1 \phi := (\phi, 0).$$

We have the following

Theorem 3.7. *1. A_Θ is a negative and injective self-adjoint operator which coincides with A on the set $\text{Ker}(\tau)$. Its resolvent is given by*

$$(-A_\Theta + \lambda^2)^{-1} = R_0(\lambda) + G(\lambda)(\Theta + \lambda^2 \tau \bar{B}^{-1} B^{-1} G(\lambda))^{-1} \check{G}(\lambda).$$

The positive quadratic form \mathcal{Q}_Θ corresponding to $-A_\Theta$ is

$$\mathcal{Q}_\Theta : \mathcal{K}_1 \subseteq \mathcal{H}_0 \rightarrow \mathbb{R}, \quad \mathcal{Q}_\Theta(\phi) = \|\bar{B}\phi_0\|_0^2 + \|\zeta_\phi\|_\Theta^2.$$

2. The skew-adjoint operator corresponding to the abstract wave equation $\ddot{\phi} = A_\Theta \phi$ is the skew-adjoint operator W_Θ defined in the previous theorem.

Proof. 1. Let us define

$$R_\Theta(\lambda) := R_0(\lambda) + G(\lambda)(-\lambda\Gamma_\Theta(\lambda))^{-1}\check{G}(\lambda).$$

By the proof of Lemma 3.2 and [11], Lemma 2.1,

$$\begin{aligned} & -\lambda\Gamma_\Theta(\lambda) - (-\mu\Gamma_\Theta(\mu)) \\ & = \text{s-}\lim_{\epsilon \downarrow 0} (\lambda^2 - \epsilon^2) \check{G}(\lambda)G(\epsilon) - (\mu^2 - \epsilon^2) \check{G}(\mu)G(\epsilon) \\ & = \tau(G(\mu) - G(\lambda)) = (\lambda^2 - \mu^2)\check{G}(\mu)\check{G}(\lambda). \end{aligned}$$

We already know that $-\lambda\Gamma_\Theta(\lambda)$ is boundedly invertible and, by (3.5) and Lemma 3.2, $(-\lambda\Gamma_\Theta(\lambda))^* = -\lambda\Gamma_\Theta(\lambda)$. Therefore, by [11], Proposition 2.1, $R_\Theta(\lambda)$ is the resolvent of a self-adjoint operator \tilde{A}_Θ , coinciding with A on $\text{Ker}(\tau)$, defined by

$$\begin{aligned} D(\tilde{A}_\Theta) &:= \{\phi \in \mathcal{H}_0 : \phi = \phi_\lambda + G(\lambda)(-\lambda\Gamma_\Theta(\lambda))^{-1}\tau\phi_\lambda\} \\ (-\tilde{A}_\Theta + \lambda^2)\phi &:= (-A + \lambda^2)\phi_\lambda. \end{aligned}$$

One then proves that $\tilde{A}_\Theta \equiv A_\Theta$ proceeding exactly as in the proof of [12], Theorem 5.

Since A is injective, $A_\Theta\phi = 0$ implies $\phi_0 = 0$ and thus $\zeta_\phi = 0$, i.e. $\phi = 0$.

By the proof of Lemma 3.1 one has

$$\langle B\phi_0, \bar{B}\bar{G}(\lambda)\zeta \rangle_0 = \langle \check{G}(\lambda)\phi_0, \zeta \rangle_{\mathfrak{h}}$$

and, by (2.2),

$$\langle B\bar{B}\phi_0, G(\lambda)\zeta_\phi \rangle_0 + \lambda^2[\phi_0, \bar{G}(\lambda)\zeta_\phi]_1 = \langle \tau\phi_0, \zeta_\phi \rangle_{\mathfrak{h}}.$$

Thus, using the definition of G and the two different decompositions of $\phi \in D(A_\Theta)$ given by

$$\phi = \phi_0 + G\zeta_\phi = \phi_\lambda + G(\lambda)\zeta_\phi,$$

one obtains

$$\begin{aligned} \langle -A_\Theta\phi, \phi \rangle_0 &= \langle -\bar{A}\phi_0, \phi_\lambda \rangle_0 + \langle -\bar{A}\phi_0, G(\lambda)\zeta_\phi \rangle_0 \\ &= \langle \bar{B}\phi_0, \bar{B}\phi_\lambda \rangle_0 + \langle \bar{B}B\phi_0, G(\lambda)\zeta_\phi \rangle_0 \\ &= \langle \bar{B}\phi_0, \bar{B}\phi_0 \rangle_0 + \lambda^2\langle \bar{B}\phi_0, \bar{B}G(\lambda)\zeta_\phi \rangle_0 - \lambda^2\langle \bar{B}\phi_0, \bar{B}G(\lambda)\zeta_\phi \rangle_0 + \langle \tau\phi_0, \zeta_\phi \rangle_{\mathfrak{h}} \\ &= \langle \bar{B}\phi_0, \bar{B}\phi_0 \rangle_0 + \langle \zeta_\phi, \zeta_\phi \rangle_\Theta. \end{aligned}$$

Thus A_Θ is negative. Since \mathcal{K}_1 is obviously complete with respect to the norm

$$\|\phi\|_{\mathcal{K}_1}^2 := \|\bar{B}\phi_0\|_0^2 + \|\zeta_\phi\|_\Theta^2 + \|\phi\|_0^2,$$

the closed and positive quadratic form \mathcal{Q}_Θ is the one associated to $-A_\Theta$.

2. Since the completion of \mathcal{K}_1 with respect to the scalar product

$$[\phi, \varphi]_1 := \langle B\phi_0, B\varphi_0 \rangle + \langle \zeta_\phi, \zeta_\varphi \rangle_\Theta$$

is $\bar{\mathcal{K}}_1$ and the completion of $D(A_\Theta)$ with respect to the scalar product

$$[\phi, \varphi]_2 := \langle B\phi_0, B\varphi_0 \rangle + \langle \zeta_\phi, \zeta_\varphi \rangle_\Theta + \langle A\phi_0, A\varphi_0 \rangle$$

is $\{\phi \in \bar{\mathcal{K}}_2 : \Theta\zeta_\phi = \tau\phi_0\}$, one has that $\bar{A}_\Theta\phi = \bar{A}\phi_0$ for any ϕ in such a set and the proof is done. \square

4. SINGULAR PERTURBATIONS OF GENERALIZED ABSTRACT WAVE EQUATIONS

In this section we look for singular perturbations of operators of the kind

$$W_g(\phi, \psi) := (\bar{C}_2\phi + \psi, C_1\psi + \bar{A}\phi).$$

Let us begin with the simpler case in which

$$W_g(\phi, \varphi) := (\varphi, C\varphi + \bar{A}\phi),$$

where

$$C : \mathcal{H}_1 \subseteq \mathcal{H}_0 \rightarrow \mathcal{H}_0$$

is a skew-adjoint operator such that:

H4.1)

$$\forall \phi \in \mathcal{H}_1, \quad \|C\phi\|_0 \leq c \|B\phi\|_0;$$

H4.2)

$$C(\mathcal{H}_2) \subseteq \mathcal{H}_1 \quad \text{and} \quad \forall \phi \in \mathcal{H}_2, \quad BC\phi = CB\phi.$$

Lemma 4.1. *If H4.1 and H4.2 hold true then*

$$B^2 - \lambda C + \lambda^2 : \mathcal{H}_2 \subseteq \mathcal{H}_0 \rightarrow \mathcal{H}_0$$

is invertible for all $\lambda \neq 0$,

$$R(\lambda) := (B^2 - \lambda C + \lambda^2)^{-1} \in \mathcal{B}(\mathcal{H}_0, \mathcal{H}_2).$$

and

$$\forall \phi \in \mathcal{H}_1, \quad \|B(B^2 + \lambda^2)R(\lambda)\phi\|_0 \leq c \|B\phi\|_0.$$

Proof. By our hypotheses one has

$$\forall \phi \in \mathcal{H}_2, \quad \|BC\phi\|_0 = \|CB\phi\|_0 \leq c \|B^2\phi\|_0.$$

Thus, by induction,

$$\forall k \geq 1, \quad \forall \phi \in \mathcal{H}_{k+1}, \quad \|B^k C\phi\|_0 \leq c \|B^{k+1}\phi\|_0,$$

and $C(\mathcal{H}_{k+1}) \subseteq \mathcal{H}_k$ for any $k \geq 1$. By

$$\forall \phi \in \mathcal{H}_3, \quad B^2 C\phi = BCB\phi = CB^2\phi$$

one gets

$$\forall \phi \in \mathcal{H}_1, \quad R_0(\lambda)C\phi = CR_0(\lambda)\phi,$$

so that $CR_0(\lambda)$ is skew-adjoint. Thus $1 - \lambda CR_0(\lambda)$ is boundedly invertible for all $\lambda \neq 0$ and

$$R(\lambda) = (1 - \lambda CR_0(\lambda))^{-1}R_0(\lambda) = R_0(\lambda)(1 - \lambda CR_0(\lambda))^{-1}.$$

This gives

$$\begin{aligned} \|(B^2 + \lambda^2)R(\lambda)\phi\|_0 &= \|(1 - \lambda CR_0(\lambda))^{-1}\phi\|_0 \\ &\leq \|(1 - \lambda CR_0(\lambda))^{-1}\|_{\mathcal{H}_0, \mathcal{H}_0} \|\phi\|_0 \end{aligned}$$

and

$$\begin{aligned} \|B(B^2 + \lambda^2)R(\lambda)\phi\|_0 &= \|B(1 - \lambda CR_0(\lambda))^{-1}\phi\|_0 \\ &= \|(1 - \lambda CR_0(\lambda))^{-1}B\phi\|_0 \leq \|(1 - \lambda CR_0(\lambda))^{-1}\|_{\mathcal{H}_0, \mathcal{H}_0} \|B\phi\|_0. \end{aligned}$$

□

Let $\bar{C} \in \mathcal{B}(\bar{\mathcal{H}}_1, \mathcal{H}_0)$ be the closed bounded extension of operator

$$C : \mathcal{H}_1 \subseteq \bar{\mathcal{H}}_1 \rightarrow \mathcal{H}_0.$$

It exists by H4.1. Let $\bar{R}(\lambda) \in \mathcal{B}(\bar{\mathcal{H}}_1, \bar{\mathcal{H}}_3)$ the closed bounded extension of

$$R(\lambda) : \mathcal{H}_1 \subseteq \bar{\mathcal{H}}_1 \rightarrow \bar{\mathcal{H}}_3.$$

It exists by Lemma 4.1. For such an extension the following relations hold true:

$$\begin{aligned} (-\bar{A} - \lambda \bar{C})\bar{R}(\lambda) + \lambda^2 \bar{R}(\lambda) &= 1_{\bar{\mathcal{H}}_1}, \\ R(\lambda)(-\bar{A} - \lambda \bar{C}) + \lambda^2 \bar{R}(\lambda) &= 1_{\bar{\mathcal{H}}_2}. \end{aligned}$$

Proceeding as in theorem 2.5 one obtains the following

Theorem 4.2. *Under hypotheses H4.1 and H4.2 the linear operator*

$$W_g : \bar{\mathcal{H}}_2 \times \mathcal{H}_1 \subseteq \bar{\mathcal{H}}_1 \oplus \mathcal{H}_0 \rightarrow \bar{\mathcal{H}}_1 \oplus \mathcal{H}_0, \quad W_g(\phi, \varphi) := (\varphi, C\varphi + \bar{A}\phi),$$

is skew-adjoint and its resolvent is given by

$$\begin{aligned} &(-W_g + \lambda)^{-1}(\phi, \varphi) \\ &= (\lambda \bar{R}(\lambda)\phi + R(\lambda)(-\bar{C}\phi + \varphi), -\phi + \lambda^2 \bar{R}(\lambda)\phi + \lambda R(\lambda)(-\bar{C}\phi + \varphi)). \end{aligned}$$

Remark 4.3. We used the notation $\bar{\mathcal{H}}_2 \times \mathcal{H}_1$ for $D(W)$ since, when $C \neq 0$, the scalar product inducing the graph norm on $D(W_g)$ is different from the one of $\bar{\mathcal{H}}_2 \oplus \mathcal{H}_1$.

By the previous theorem

$$\tilde{W}_g : \bar{\mathcal{H}}_2 \times \mathcal{H}_1 \times \mathfrak{h}_\Theta \subseteq \bar{\mathcal{H}}_1 \oplus \mathcal{H}_0 \oplus \mathfrak{h}_\Theta \rightarrow \bar{\mathcal{H}}_1 \oplus \mathcal{H}_0 \oplus \mathfrak{h}_\Theta ,$$

$$\tilde{W}_g(\phi, \varphi, \zeta) := (W_g(\psi, \varphi), 0)$$

is skew-adjoint and $(-\tilde{W}_g + \lambda)^{-1}(\phi, \varphi, \zeta) = ((-W_g + \lambda)^{-1}(\phi, \varphi), \lambda^{-1}\zeta)$.

Now we consider a sequence $J_\nu : \mathcal{H}_0 \rightarrow \mathcal{H}_0$, $\nu > 0$, of self-adjoint operators such that

1.

$$J_\nu \in \mathcal{B}(\mathcal{H}_k, \mathcal{H}_{k+1}), \quad k \geq 0;$$

2.

$$\forall \phi \in \mathcal{H}_1 \quad J_\nu B\phi = BJ_\nu\phi \quad J_\nu C\phi = CJ_\nu\phi;$$

3.

$$\forall \phi \in \mathcal{H}_0, \quad \lim_{\nu \downarrow 0} \|J_\nu\phi - \phi\|_0 = 0.$$

Such sequence J_ν can be obtained by considering, for example, the family $(\nu B^2 + 1)^{-1}$, but other choices are possible (see Example 3 in the next section). We remark that the successive construction will depend on the choice we make for such a family.

Denoting by $\bar{J}_\nu \in \mathcal{B}(\bar{\mathcal{H}}_k, \bar{\mathcal{H}}_{k+1})$, $k \geq 1$, the closed bounded extension of J_ν and given $\tau \in \mathcal{B}(\bar{\mathcal{H}}_2, \mathfrak{h})$ we define the bounded linear map

$$\tau_\nu := \tau \bar{J}_\nu : \bar{\mathcal{H}}_1 \rightarrow \mathfrak{h},$$

and

$$D(\bar{\tau}) := \{\phi \in \bar{\mathcal{H}}_1 : \lim_{\nu \downarrow 0} \tau_\nu\phi \text{ exists in } \mathfrak{h}\},$$

$$\bar{\tau} : D(\bar{\tau}) \subseteq \bar{\mathcal{H}}_1 \rightarrow \mathfrak{h}, \quad \bar{\tau}\phi := \lim_{\nu \downarrow 0} \tau_\nu\phi.$$

Note that for all $\phi \in \bar{\mathcal{H}}_2$, by 3,

$$\begin{aligned} & \lim_{\nu \downarrow 0} \|B\bar{B}(\bar{J}_\nu\phi - \phi)\|_0^2 + \|\bar{B}(\bar{J}_\nu\phi - \phi)\|_0^2 \\ &= \lim_{\nu \downarrow 0} \|J_\nu B\bar{B}\phi - B\bar{B}\phi\|_0^2 + \|J_\nu \bar{B}\phi - \bar{B}\phi\|_0^2 = 0, \end{aligned}$$

so that $\bar{\mathcal{H}}_2 \subseteq D(\bar{\tau})$ and $\bar{\tau} = \tau$ on $\bar{\mathcal{H}}_2$.

Defining then

$$\tau_\Theta : D(\bar{\tau}) \times \mathcal{H}_0 \times \mathfrak{h}_\Theta \rightarrow \mathfrak{h}, \quad \tau_\Theta(\phi, \psi, \zeta) := \bar{\tau}\phi - \Theta\zeta$$

we have that τ_Θ satisfies A.1 and A.2.

Now we define $\check{G}(\lambda) \in \mathcal{B}(\mathcal{H}_0, \mathfrak{h})$ and $G(\lambda) \in \mathcal{B}(\mathfrak{h}, \mathcal{H}_0)$ by

$$\check{G}(\lambda) := \tau R(\lambda), \quad G(\lambda) := \check{G}(-\lambda)^*.$$

We also define $\check{\bar{G}}(\lambda) \in \mathcal{B}(\bar{\mathcal{H}}_1, \mathfrak{h})$ and $\bar{G}(\lambda) \in \mathcal{B}(\mathfrak{h}, \bar{\mathcal{H}}_1)$ by

$$\check{\bar{G}}(\lambda) := \tau \bar{R}(\lambda), \quad \bar{G}(\lambda) := \check{\bar{G}}(-\lambda)^*.$$

Obviously $\check{G}(\lambda) = \check{G}(\lambda)$ on \mathcal{H}_1 . As in the previous section one has the following

Lemma 4.4.

$$\begin{aligned} \text{Ran}(G(\lambda)) &\subseteq \text{Ran}(B) \quad \text{and} \quad \bar{G}(\lambda) = \bar{B}^{-1}B^{-1}G(\lambda). \\ \text{Ran}(\bar{G}(\lambda)) &\subseteq \bar{\mathcal{H}}_2 \quad \text{and} \quad \bar{G}(\lambda) \in \mathcal{B}(\mathfrak{h}, \bar{\mathcal{H}}_2). \end{aligned}$$

Now we define $\check{G}_\Theta(\lambda) \in \mathcal{B}(\bar{\mathcal{H}}_1 \oplus \mathcal{H}_0 \oplus \mathfrak{h}_\Theta, \mathfrak{h})$ by

$$(4.1) \quad \check{G}_\Theta(\lambda)(\phi, \varphi, \zeta) := \tau_\Theta(-\tilde{W} + \lambda)^{-1}(\phi, \varphi, \zeta)$$

$$(4.2) \quad = \lambda \check{G}(\lambda)\phi + \check{G}(\lambda)(-\bar{C}\phi + \varphi) - \lambda^{-1}\Theta\zeta$$

and $G_\Theta(\lambda) \in \mathcal{B}(\mathfrak{h}, \bar{\mathcal{H}}_1 \oplus \mathcal{H}_0 \oplus \mathfrak{h}_\Theta)$ by

$$(4.3) \quad G_\Theta(\lambda)\zeta := -\check{G}_\Theta(-\lambda)^*\zeta$$

$$(4.4) \quad = (\lambda \bar{G}(\lambda)\zeta + \bar{C}^*G(\lambda)\zeta, -G(\lambda)\zeta, -\lambda^{-1}\zeta).$$

Regarding the adjoint of \bar{C} one has the following

Lemma 4.5.

$$\bar{C}^* = -\bar{B}^{-1}\bar{C}\bar{B}^{-1}.$$

Proof. Since C commutes with B , for any ϕ in \mathcal{H}_1 one has

$$\bar{C}\bar{B}^{-1}\phi = \bar{B}^{-1}C\phi,$$

and thus for any ϕ and ψ in \mathcal{H}_1 one has

$$\begin{aligned} [-\bar{B}^{-1}\bar{C}\bar{B}^{-1}\phi, \psi]_1 &= -\langle \bar{C}\bar{B}^{-1}\phi, B\psi \rangle_0 \\ &= -\langle \bar{B}^{-1}C\phi, B\psi \rangle_0 = -\langle C\phi, \psi \rangle_0 = \langle \phi, C\psi \rangle_0. \end{aligned}$$

□

Let us now consider the bounded linear map

$$\Gamma_{\nu, \Theta}(\lambda) := -\tau_\nu \bar{B}^{-1}(-C + \lambda)B^{-1}G(\lambda) - \frac{1}{\lambda} \Theta.$$

We have the following

Lemma 4.6.

$$\Gamma_{\nu, \Theta}(\lambda)^* = -\Gamma_{\nu, \Theta}(-\lambda).$$

Proof. At first let us observe that, being $CR_0(\epsilon)$ skew-adjoint (see the proof of Lemma 4.1), one has

$$\forall \epsilon > 0, \quad \|(1 \pm \epsilon CR_0(\epsilon))^{-1}\|_{\mathcal{H}_0, \mathcal{H}_0} \leq 1.$$

Thus, using H4.1, functional calculus and dominated convergence theorem,

$$\begin{aligned} \lim_{\epsilon \downarrow 0} \|((1 \pm \epsilon C R_0(\epsilon))^{-1} - 1)\phi\|^2 &\leq \lim_{\epsilon \downarrow 0} \|\epsilon C R_0(\epsilon)\phi\|^2 \\ &\leq c^2 \lim_{\epsilon \downarrow 0} \|\epsilon B R_0(\epsilon)\phi\|^2 = \lim_{\epsilon \downarrow 0} \int_{\mathbb{R}} d\mu_\phi(x) \left| \frac{\epsilon x}{x^2 + \epsilon^2} \right|^2 = 0. \end{aligned}$$

Since

$$B^2 R(\pm\epsilon) = B^2 R_0(\epsilon)(1 \pm \epsilon C R_0(\epsilon))^{-1},$$

and $B^2 R_0(\epsilon)$ strongly converges to $1_{\mathcal{H}_0}$ when $\epsilon \downarrow 0$ (see the proof of Lemma 3.2), one has that

$$\text{s-}\lim_{\epsilon \downarrow 0} B^2 R(\pm\epsilon) = 1_{\mathcal{H}_0}.$$

This implies (proceeding as in the proof of Lemma 4.1) that $R(\epsilon)G(\lambda)$ strongly converges in $\mathcal{B}(\mathfrak{h}, \mathcal{H}_2)$ to $\bar{B}^{-1}B^{-1}G(\lambda)$ when $\epsilon \downarrow 0$. Therefore

$$\begin{aligned} &\Gamma_{\nu, \Theta}(\lambda) \\ &= \text{s-}\lim_{\epsilon \downarrow 0} -\frac{1}{\lambda} \left(\Theta + \lambda \tau_\nu(-C + \lambda \pm \epsilon) (B^2 \mp \epsilon C + \epsilon^2)^{-1} G(\lambda) \right), \end{aligned}$$

and the proof is concluded by showing that

$$\begin{aligned} &(\tau_\nu C (B^2 - \epsilon C + \epsilon^2)^{-1} G(\lambda))^* \\ &= -\tau_\nu C (B^2 + \epsilon C + \epsilon^2)^{-1} G(-\lambda). \end{aligned}$$

Proceeding as in [11], Lemma 2.1, by first resolvent identity one obtains

$$\begin{aligned} &(-C + \lambda + \epsilon) (B^2 - \epsilon C + \epsilon^2)^{-1} G(\lambda) \\ &= \frac{G(\epsilon) - G(\lambda)}{\lambda - \epsilon} \\ &= (-C + \lambda + \epsilon) (B^2 - \lambda C + \lambda^2)^{-1} G(\epsilon), \end{aligned}$$

so that

$$(B^2 - \epsilon C + \epsilon^2)^{-1} G(\lambda) = (B^2 - \lambda C + \lambda^2)^{-1} G(\epsilon).$$

Therefore we need to show that

$$\begin{aligned} &(\tau_\nu C (B^2 - \lambda C + \lambda^2)^{-1} G(\epsilon))^* \\ &= -\tau_\nu C (B^2 + \epsilon C + \epsilon^2)^{-1} G(-\lambda). \end{aligned}$$

Since C , B and J_ν commute, we have

$$\begin{aligned} &(\tau_\nu C (B^2 - \lambda C + \lambda^2)^{-1} G(\epsilon))^* = (\check{G}(\lambda) C J_\nu G(\epsilon))^* \\ &= G(\epsilon)^* (C J_\nu)^* \check{G}(\lambda)^* = -\check{G}(-\epsilon) C J_\nu G(-\lambda) \\ &= -\tau_\nu C (B^2 + \epsilon C + \epsilon^2)^{-1} G(-\lambda) \end{aligned}$$

and the proof is done. \square

Now we suppose that
H4.3)

$$\text{Ran}(\bar{C}^*G(\lambda)) \equiv \text{Ran}(\bar{C}\bar{G}(\lambda)) \subseteq D(\bar{\tau}).$$

Note that H4.3 is always verified if $\bar{C} \in \mathcal{B}(\bar{\mathcal{H}}_2, \bar{\mathcal{H}}_2)$, but such a hypothesis can hold true also in situations where C is unbounded (see Example 3 in the next section). Then, by the uniform boundedness principle $\bar{\tau}\bar{C}^*G(\lambda) \in \mathcal{B}(\mathfrak{h})$, so that A.4 is satisfied,

$$(4.5) \quad \Gamma_\Theta(\lambda) := -\tau_\Theta G_\Theta(\lambda) = -\lambda\tau\bar{B}^{-1}B^{-1}G(\lambda) - \bar{\tau}\bar{C}^*G(\lambda) - \frac{1}{\lambda}\Theta$$

$$(4.6) \quad = -\bar{\tau}\bar{B}^{-1}(-C + \lambda)B^{-1}G(\lambda) - \frac{1}{\lambda}\Theta$$

is well-defined and $\Gamma_\Theta(\lambda) \in \mathcal{B}(\mathfrak{h})$. By H4.3 one has

$$\Gamma_\Theta(\lambda) = \text{s-}\lim_{\epsilon \downarrow 0} \Gamma_{\nu, \Theta}(\lambda),$$

so that the previous lemma implies

$$\Gamma_\Theta(\lambda)^* = -\Gamma_\Theta(-\lambda).$$

Thus A.5 is satisfied. Suppose now that H3.1 holds true. Then, since

$$R(\lambda) = R_0(\lambda)(1 - \lambda C R_0(\lambda))^{-1},$$

one has

$$\text{Ran}(G(\lambda)) \cap \mathcal{H}_1 = \{0\},$$

so that A.3 is satisfied. In conclusion, by Theorem 6.2 we can define a skew-adjoint extension of the skew-symmetric operator given by restricting \tilde{W} to the dense set

$$\mathcal{N}_\Theta := \{(\phi, \varphi, \zeta) \in \bar{\mathcal{H}}_2 \times \mathcal{H}_1 \times \mathfrak{h}_\Theta : \tau\phi = \Theta\zeta\} :$$

Theorem 4.7. *Suppose that H3.0, H3.1, H4.1, H4.2 and H4.3 hold true. Let*

$$\begin{aligned} D(\tilde{W}_\Theta) \\ := \{(\phi_0, \varphi_0, \zeta_\phi) \in \bar{\mathcal{H}}_1 \oplus \mathcal{H}_0 \oplus \mathfrak{h}_\Theta : \phi_0 = \phi_\lambda + \bar{B}^{-1}CB^{-1}G(\lambda)\zeta_\phi, \\ \varphi_0 = \varphi_\lambda + G(\lambda)\zeta_\phi, \phi_\lambda \in \bar{\mathcal{H}}_2, \varphi_\lambda \in \mathcal{H}_1, \zeta_\phi \in \mathfrak{h}, \Theta\zeta_\phi = \bar{\tau}\phi_0\} . \end{aligned}$$

Then

$$\tilde{W}_\Theta : D(\tilde{W}_\Theta) \subseteq \bar{\mathcal{H}}_1 \oplus \mathcal{H}_0 \oplus \mathfrak{h}_\Theta \rightarrow \bar{\mathcal{H}}_1 \oplus \mathcal{H}_0 \oplus \mathfrak{h}_\Theta,$$

$$\tilde{W}_\Theta(\phi_0, \varphi_0, \zeta_\phi) := (\varphi_\lambda - \lambda\bar{B}^{-1}(-C + \lambda)B^{-1}G(\lambda)\zeta_\phi, C\varphi_\lambda + \bar{A}\phi_\lambda, \zeta_\phi)$$

is a skew-adjoint extension of the restriction of

$$\tilde{W}_g : \bar{\mathcal{H}}_2 \times \mathcal{H}_1 \times \mathfrak{h}_\Theta \subseteq \bar{\mathcal{H}}_1 \oplus \mathcal{H}_0 \oplus \mathfrak{h}_\Theta \rightarrow \bar{\mathcal{H}}_1 \oplus \mathcal{H}_0 \oplus \mathfrak{h}_\Theta,$$

$$\tilde{W}_g(\phi, \psi, \zeta) = (\psi, C\phi + \bar{A}\phi, 0)$$

to the dense set \mathcal{N}_Θ . The resolvent of \tilde{W}_Θ is given by

$$(-\tilde{W}_\Theta + \lambda)^{-1} = (-\tilde{W}_g + \lambda)^{-1} + G_\Theta(\lambda)\Gamma_\Theta(\lambda)^{-1}\check{G}_\Theta(\lambda),$$

where the linear operators $(-\tilde{W} + \lambda)^{-1}$, $\check{G}_\Theta(\lambda)$, $G_\Theta(\lambda)$, $\Gamma_\Theta(\lambda)$, have been defined in Theorem 4.2, (4.2), (4.4) and (4.6) respectively.

Proof. By Theorem 6.4 we known that $(-\tilde{W}_g + \lambda)^{-1} + G_\Theta(\lambda)\Gamma_\Theta(\lambda)^{-1}\check{G}_\Theta(\lambda)$ is the resolvent of a skew-adjoint extension \hat{W}_Θ of the restriction of \tilde{W}_g to the dense set \mathcal{N}_Θ . Therefore $(\hat{\phi}_0, \hat{\phi}, \hat{\zeta}_\phi) \in D(\hat{W}_\Theta)$ if and only if

$$\begin{aligned}\hat{\phi}_0 &= \hat{\phi}_\lambda + (\lambda\bar{B}^{-1}B^{-1} + \bar{C}^*)G(\lambda)\Gamma_\Theta(\lambda)^{-1}(\tau\hat{\phi}_\lambda - \Theta\zeta_\lambda), \\ \hat{\phi} &= \varphi_\lambda - G(\lambda)\Gamma_\Theta(\lambda)^{-1}(\tau\hat{\phi}_\lambda - \Theta\zeta_\lambda), \\ \hat{\zeta}_\phi &= \zeta_\lambda - \frac{1}{\lambda}\Gamma_\Theta(\lambda)^{-1}(\tau\hat{\phi}_\lambda - \Theta\zeta_\lambda),\end{aligned}$$

where

$$\hat{\phi}_\lambda \in \bar{\mathcal{H}}_2, \quad \varphi_\lambda \in \mathcal{H}_1, \quad \zeta_\lambda \in \mathfrak{h}.$$

Let us now show that $D(\hat{W}_\Theta) = D(\tilde{W}_\Theta)$.

Since $\text{Ran}(\bar{G}(\lambda)) \subseteq \bar{\mathcal{H}}_2$, so that

$$\hat{\phi}_\lambda + \lambda\bar{B}^{-1}B^{-1}G(\lambda)\Gamma_\Theta(\lambda)^{-1}(\tau\hat{\phi}_\lambda - \Theta\zeta_\lambda) \in \bar{\mathcal{H}}_2,$$

and

$$\begin{aligned}\tau_\Theta((-\hat{W}_g + \lambda)^{-1} + G_\Theta(\lambda)\Gamma_\Theta(\lambda)^{-1}\check{G}_\Theta(\lambda)) \\ = \check{G}_\Theta(\lambda) - \Gamma_\Theta(\lambda)\Gamma_\Theta(\lambda)^{-1}\check{G}_\Theta(\lambda) = 0,\end{aligned}$$

so that $\tau\hat{\phi}_0 = \Theta\hat{\zeta}_\phi$, we have $D(\hat{W}_\Theta) \subseteq D(\tilde{W}_\Theta)$. Let us now prove the reverse inclusion. Given $(\phi_0, \varphi_0, \zeta_\phi) \in D(\tilde{W}_\Theta)$ let us define

$$\hat{\phi}_\lambda := \phi_\lambda + \lambda\bar{G}(\lambda)\zeta_\varphi, \quad \zeta_\lambda := \zeta_\phi - \frac{1}{\lambda}\zeta_\varphi.$$

Then

$$\bar{\tau}\phi_0 = \tau\hat{\phi}_\lambda - \lambda\tau\bar{G}(\lambda)\zeta_\varphi - \bar{\tau}\bar{C}^*G(\lambda)\zeta_\varphi = \Theta\zeta_\phi = \Theta\left(\zeta_\lambda + \frac{1}{\lambda}\zeta_\varphi\right)$$

implies

$$\tau\hat{\phi}_\lambda - \Theta\zeta_\lambda = \left(\lambda\tau\bar{G}(\lambda) + \bar{\tau}\bar{C}^*G(\lambda) + \frac{1}{\lambda}\Theta\right)\zeta_\varphi,$$

i.e.

$$\zeta_\varphi = -\Gamma_\Theta(\lambda)^{-1}(\tau\hat{\phi}_\lambda - \Theta\zeta_\lambda).$$

Thus $D(\tilde{W}_\Theta) \subseteq D(\hat{W}_\Theta)$. Now we have

$$\begin{aligned}
\hat{W}_\Theta(\phi_0, \psi, \zeta_\phi) &= \tilde{W}_g(\hat{\phi}_\lambda, \varphi_\lambda, \zeta_\lambda) + \lambda(\phi_0 - \hat{\phi}_\lambda, \varphi - \varphi_\lambda, \zeta_\phi - \zeta_\lambda) \\
&= (\varphi_\lambda - \lambda^2 \bar{G}(\lambda) \zeta_\varphi - \lambda \bar{C}^* G(\lambda) \zeta_\varphi, C\varphi_\lambda + \bar{A} \hat{\phi}_\lambda + \lambda G(\lambda) \zeta_\varphi, \zeta_\varphi) \\
&= (\varphi_\lambda - \lambda^2 \bar{G}(\lambda) \zeta_\varphi - \lambda \bar{C}^* G(\lambda) \zeta_\varphi, C\varphi_\lambda + \bar{A}(\phi_\lambda + \lambda \bar{G}(\lambda) \zeta_\varphi) + \lambda G(\lambda) \zeta_\varphi, \zeta_\varphi) \\
&= (\varphi_\lambda - \lambda(\lambda \bar{B}^{-1} B^{-1} - \bar{B}^{-1} C B^{-1}) G(\lambda) \zeta_\varphi, C\varphi_\lambda + \bar{A} \phi_\lambda, \zeta_\varphi) \\
&= (\varphi_\lambda - \lambda \bar{B}^{-1} (-C + \lambda) B^{-1} G(\lambda) \zeta_\varphi, C\varphi_\lambda + \bar{A} \phi_\lambda, \zeta_\varphi) \\
&= \tilde{W}_\Theta(\phi_0, \psi, \zeta_\phi).
\end{aligned}$$

□

Let us now consider two skew-adjoint operators

$$C_1 : \mathcal{H}_1 \subseteq \mathcal{H}_0 \rightarrow \mathcal{H}_0, \quad C_2 : \mathcal{H}_1 \subseteq \mathcal{H}_0 \rightarrow \mathcal{H}_0,$$

such that
H4.1.1)

$$\forall \phi \in \mathcal{H}_1, \quad \|C_1 \phi\|_0 \leq c_1 \|B\phi\|_0, \quad \|C_2 \phi\|_0 \leq c_2 \|B\phi\|_0, \quad c_1 c_2 < 1$$

H4.2.1)

$$C_1(\mathcal{H}_2) \subseteq \mathcal{H}_1, \quad C_2(\mathcal{H}_2) \subseteq \mathcal{H}_1.$$

and

$$\forall \phi \in \mathcal{H}_2, \quad C_1 C_2 \phi = C_2 C_1 \phi, \quad B C_1 \phi = C_1 B \phi, \quad B C_2 \phi = C_2 B \phi.$$

Then by the Kato-Rellich theorem

$$-A_C := B^2 + C_1 C_2 : \mathcal{H}_2 \subseteq \mathcal{H}_0 \rightarrow \mathcal{H}_0$$

is self-adjoint, positive and injective. Let B_C be the self-adjoint, positive and injective operator defined by $B_C := (-A_C)^{1/2}$. Since, by H4.2.1,

$$(1 - c_1 c_2) \|B\phi\| \leq \|B_C \phi\| \leq (1 + c_1 c_2) \|B\phi\|,$$

the domain of B_C coincides with the space \mathcal{H}_1 , the domain of B . Moreover, since B and B_C commutes,

$$(1 - c_1 c_2)^k \|B^k \phi\| \leq \|B_C^k \phi\| \leq (1 + c_1 c_2)^k \|B^k \phi\|,$$

thus the Hilbert spaces generated by B_C coincide, as Banach spaces (in the sense that each space has an equivalent norm), with the ones generated by B , i.e. coincide with \mathcal{H}_k , $\bar{\mathcal{H}}_k$, and \mathcal{H}_{-k} , $k \geq 1$.

Let $\bar{A}_C := -B_C \bar{B}_C \in \mathcal{B}(\bar{\mathcal{H}}_2, \mathcal{H}_0)$, where $\bar{B}_C \in \mathcal{B}(\bar{\mathcal{H}}_1, \mathcal{H}_0)$ is the closed bounded extension of $B_C : \mathcal{H}_1 \subseteq \bar{\mathcal{H}}_1 \rightarrow \mathcal{H}_0$. We know that \bar{A}_C coincides with the closed bounded extension of $A_C : \mathcal{H}_2 \subseteq \bar{\mathcal{H}}_2 \rightarrow \mathcal{H}_0$. Since C_2 commutes with B , by H4.2.1 we have

$$\|B C_2 \phi\|_0 \leq c_2 \|B^2 \phi\|_0.$$

Thus we can define $\bar{C}_2 \in \mathbf{B}(\bar{\mathcal{H}}_1, \mathcal{H}_0) \cap \mathbf{B}(\bar{\mathcal{H}}_2, \mathcal{H}_1)$ as the closed bounded extension of $C_2 : \mathcal{H}_1 \subseteq \bar{\mathcal{H}}_1 \rightarrow \mathcal{H}_0$ and

$$\bar{A}_C = \bar{A} - C_1 \bar{C}_2.$$

Since $C := C_1 + C_2$ and $B_C := \sqrt{\bar{B}^2 + C_1 \bar{C}_2}$ satisfy H4.1 and H4.2, by Theorem 4.2 we have that

$$W_g : \bar{\mathcal{H}}_2 \times \mathcal{H}_1 \subseteq \bar{\mathcal{H}}_1 \oplus \mathcal{H}_0 \rightarrow \bar{\mathcal{H}}_1 \oplus \mathcal{H}_0$$

$$W_g(\phi, \varphi) := (\varphi, (C_1 + C_2)\varphi + (\bar{A} - C_1 \bar{C}_2)\phi)$$

is skew-adjoint once we put on $\bar{\mathcal{H}}_1 \oplus \mathcal{H}_0$ the scalar product

$$\langle\langle (\phi_1, \varphi_1), (\phi_2, \varphi_2) \rangle\rangle := \langle \bar{B}_C \phi_1, \bar{B}_C \phi_2 \rangle_0 + \langle \varphi_1, \varphi_2 \rangle_0.$$

Let us define the Hilbert space $(\mathcal{H}_C, \langle\langle \cdot, \cdot \rangle\rangle_C)$ by $\mathcal{H}_C = \bar{\mathcal{H}}_1 \times \mathcal{H}_0$,

$$\begin{aligned} & \langle\langle (\phi_1, \psi_1), (\phi_2, \psi_2) \rangle\rangle_C \\ &:= \langle \bar{B}_C \phi_1, \bar{B}_C \phi_2 \rangle_0 + \langle \psi_1 + \bar{C}_2 \phi_1, \psi_2 + \bar{C}_2 \phi_2 \rangle_0 \\ &= \langle \bar{B}_C \phi_1, \bar{B}_C \phi_2 \rangle_0 + \langle \bar{C}_2 \phi_1, \psi_2 \rangle_0 + \langle \psi_1, \bar{C}_2 \phi_2 \rangle_0 + \langle \psi_1, \psi_2 \rangle_0 \\ &+ \langle (\bar{C}_2 - \bar{C}_1) \phi_1, \bar{C}_2 \phi_2 \rangle_0, \end{aligned}$$

where $\bar{C}_1 \in \mathbf{B}(\bar{\mathcal{H}}_1, \mathcal{H}_0)$ denotes the closed bounded extension of

$$C_1 : \mathcal{H}_1 \subseteq \bar{\mathcal{H}}_1 \rightarrow \mathcal{H}_0.$$

Then the map

$$S : \bar{\mathcal{H}}_1 \oplus \mathcal{H}_0 \rightarrow \mathcal{H}_C, \quad S(\phi, \varphi) := (\phi, \varphi - \bar{C}_2 \phi)$$

is unitary and the linear operator

$$SW_g S^* : \bar{\mathcal{H}}_2 \times \mathcal{H}_1 \subseteq \mathcal{H}_C \rightarrow \mathcal{H}_C$$

$$\begin{aligned} SW_g S^*(\phi, \psi) &= SW_g(\phi, \psi + \bar{C}_2 \phi) \\ &= S(\bar{C}_2 \phi + \psi, (C_1 + C_2)(\psi + \bar{C}_2 \phi) + (\bar{A} - C_1 \bar{C}_2)\phi) \\ &= S(\bar{C}_2 \phi + \psi, (C_1 + C_2)\psi + (\bar{A} + C_2 \bar{C}_2)\phi) \\ &= (\bar{C}_2 \phi + \psi, (C_1 + C_2)\psi + (\bar{A} + C_2 \bar{C}_2)\phi - \bar{C}_2(\bar{C}_2 \phi + \psi)) \\ &= (\bar{C}_2 \phi + \psi, C_1 \psi + \bar{A} \phi) \end{aligned}$$

is skew-adjoint.

Let us now define, on the Hilbert space $\mathcal{H}_C \oplus \mathfrak{h}_\Theta$ with scalar product

$$\langle\langle (\phi_1, \psi_1, \zeta_1), (\phi_1, \psi_2, \zeta_2) \rangle\rangle_{C, \Theta} := \langle\langle (\phi_1, \psi_1), (\phi_1, \psi_2) \rangle\rangle_C + \langle \zeta_1, \zeta_2 \rangle_\Theta,$$

the skew-adjoint operator

$$\begin{aligned} \tilde{W}_g : \bar{\mathcal{H}}_2 \times \mathcal{H}_1 \times \mathfrak{h}_\Theta &\subseteq \mathcal{H}_C \oplus \mathfrak{h}_\Theta \rightarrow \mathcal{H}_C \oplus \mathfrak{h}_\Theta, \\ \tilde{W}_g(\phi, \psi, \zeta) &:= (\bar{C}_2 \phi + \psi, C_1 \psi + \bar{A} \phi, 0). \end{aligned}$$

Let

$$\check{G}_C(\lambda) := \tau(B^2 + (-C_1 + \lambda)(-C_2 + \lambda))^{-1}, \quad G_C(\lambda) := \check{G}_C(-\lambda)^*,$$

and suppose
H4.3.1)

$$\text{Ran}(\bar{C}_j^* G_C(\lambda)) \subseteq D(\bar{\tau})$$

where now

$$\bar{C}_j^* = -\bar{B}_C^{-1} \bar{C}_j \bar{B}_C^{-1}, \quad j = 1, 2,$$

Then by the previous theorem we obtain the following

Theorem 4.8. *Suppose H3.0, H3.1, H4.1.1, H4.2.1 and H4.3.1 hold true. Then the linear operator*

$$\tilde{W}_\Theta : D(\tilde{W}_\Theta) \subseteq \mathcal{H}_C \oplus \mathfrak{h}_\Theta \rightarrow \mathcal{H}_C \oplus \mathfrak{h}_\Theta,$$

$$\begin{aligned} D(\tilde{W}_\Theta) &:= \{(\phi_0, \psi_0, \zeta_\phi) \in \bar{\mathcal{H}}_1 \times \mathcal{H}_0 \times \mathfrak{h}_\Theta : \\ \phi_0 &= \phi_\lambda + \bar{B}_C^{-1}(C_1 + C_2)B_C^{-1}G_C(\lambda)\zeta_\psi, \\ \psi_0 &= \psi_\lambda + (1 - \bar{C}_2\bar{B}_C^{-1}(C_1 + C_2)B_C^{-1})G_C(\lambda)\zeta_\psi, \\ \phi_\lambda &\in \bar{\mathcal{H}}_2, \psi_\lambda \in \mathcal{H}_1, \zeta_\psi \in \mathfrak{h}, \Theta\zeta_\phi = \bar{\tau}\phi_0\}, \end{aligned}$$

$$\begin{aligned} &\tilde{W}_\Theta(\phi_0, \psi_0, \zeta_\phi) \\ &:= (\bar{C}_2\phi_\lambda + \psi_\lambda - \lambda\bar{B}_C^{-1}(-(C_1 + C_2) + \lambda)B_C^{-1}G_C(\lambda)\zeta_\psi, \\ &\quad C_1\psi_\lambda + \bar{A}\phi_\lambda + \lambda\bar{C}_2\bar{B}_C^{-1}(-(C_1 + C_2) + \lambda)B_C^{-1}G_C(\lambda)\zeta_\psi, \zeta_\psi) \end{aligned}$$

is a skew-adjoint extension of the restriction of

$$\tilde{W}_g : \bar{\mathcal{H}}_2 \times \mathcal{H}_1 \times \mathfrak{h}_\Theta \subseteq \mathcal{H}_C \oplus \mathfrak{h}_\Theta \rightarrow \mathcal{H}_C \oplus \mathfrak{h}_\Theta,$$

$$\tilde{W}_g(\phi, \psi, \zeta) := (\bar{C}_2\phi + \psi, C_1\psi + \bar{A}\phi, 0).$$

to the dense set \mathcal{N}_Θ .

Let us now suppose that H3.2 holds true. Then we can define

$$G_C : \mathfrak{h} \rightarrow \mathcal{V}, \quad G_C := G_C(\lambda) + \lambda\bar{B}_C^{-1}(-(C_1 + C_2) + \lambda)B_C^{-1}G_C(\lambda).$$

Lemma 4.9. *The definition of G_C is λ -independent. Moreover*

$$\text{Ran}(G_C) \cap \bar{\mathcal{H}}_1 = \{0\}.$$

Proof. Let $C = C_1 + C_2$. Proceeding as in [11], Lemma 2.1, by first resolvent identity one obtains

$$(\lambda - \mu)(-C + \lambda + \mu)(B_C^2 - \mu C + \mu^2)^{-1}G_C(\lambda) = G_C(\mu) - G_C(\lambda),$$

i.e.

$$\begin{aligned}
& (1 + \mu \bar{B}_C^{-1}(-C + \mu) B_C^{-1}) ((\lambda - \mu) (-C + \lambda + \mu)) \times \\
& \quad \times (B_C^2 - \mu C + \mu^2)^{-1} G_C(\lambda) \\
& = \bar{B}_C^{-1}((\lambda - \mu) (-C + \lambda + \mu)) B_C^{-1} G_C(\lambda) \\
& = (1 + \mu \bar{B}_C^{-1}(-C + \mu) B_C^{-1}) (G_C(\mu) - G_C(\lambda)) .
\end{aligned}$$

This implies

$$\begin{aligned}
& G_C(\lambda) + \lambda \bar{B}_C^{-1}(-C + \lambda) B_C^{-1} G_C(\lambda) \\
& = G_C(\mu) + \mu \bar{B}_C^{-1}(-C + \mu) B_C^{-1} G_C(\mu) .
\end{aligned}$$

Suppose there exists $\zeta \in \mathfrak{h}$ such that

$$G_C(\lambda) \zeta + \lambda \bar{B}_C^{-1}(-C + \lambda) B_C^{-1} G_C(\lambda) \zeta = \phi \in \bar{\mathcal{H}}_1 .$$

Then $G_C(\lambda) \zeta \in \mathcal{H}_1$ and so, by H3.1, $G_C(\lambda) \zeta = 0$. Thus

$$\bar{B}_C^{-1}(-C + \lambda) B_C^{-1} G_C(\lambda) \zeta = 0$$

and the proof is done. □

For any $k \geq 0$, $j = 1, 2$, let

$$\begin{aligned}
\hat{B} & : \mathcal{H}_{-k} \rightarrow \mathcal{H}_{-(k+1)} , \\
\hat{B}_C & : \mathcal{H}_{-k} \rightarrow \mathcal{H}_{-(k+1)} , \\
\hat{C}_j & : \mathcal{H}_{-k} \rightarrow \mathcal{H}_{-(k+1)} ,
\end{aligned}$$

be the closed bounded extensions of

$$\begin{aligned}
B & : \mathcal{H}_1 \subseteq \mathcal{H}_{-k} \rightarrow \mathcal{H}_{-(k+1)} , \\
B_C & : \mathcal{H}_1 \subseteq \mathcal{H}_{-k} \rightarrow \mathcal{H}_{-(k+1)}
\end{aligned}$$

and

$$\hat{C}_j : \mathcal{H}_1 \subseteq \mathcal{H}_{-k} \rightarrow \mathcal{H}_{-(k+1)} ,$$

respectively. Define also

$$\hat{A} : \bar{\mathcal{H}}_1 \rightarrow \mathcal{H}_{-1} , \quad \hat{A} := -\hat{B} \bar{B}$$

and

$$\begin{aligned}
& \hat{C}_2 G_C : \mathfrak{h} \rightarrow \mathcal{H}_{-1} , \\
& \hat{C}_2 G_C := \hat{C}_2 G_C(\lambda) + \lambda \bar{C}_2 \bar{B}_C^{-1}(-(C_1 + C_2) + \lambda) B_C^{-1} G_C(\lambda) .
\end{aligned}$$

Then

Lemma 4.10.

$$\tilde{W}_\Theta(\phi_0, \psi_0, \zeta_\phi) = (\hat{C}_2 \phi_0 + \psi_0 - G_C \zeta_\psi, \hat{C}_1 \psi_0 + \hat{A} \phi_0 + \hat{C}_2 G_C \zeta_\psi, \zeta_\psi) .$$

Proof. Since

$$\hat{B}_C C_1 = \hat{C}_1 B_C, \quad \hat{B}_C C_2 = \hat{C}_2 B_C,$$

and

$$-\hat{A} + \hat{C}_1 \bar{C}_2 = \hat{B}_C \bar{B}_C,$$

one has

$$\begin{aligned} & \bar{C}_2 \phi_\lambda + \psi_\lambda - \lambda \bar{B}_C^{-1} (-(C_1 + C_2) + \lambda) B_C^{-1} G_C(\lambda) \zeta_\psi \\ &= \hat{C}_2 \phi_0 - \bar{C}_2 \bar{B}_C^{-1} (-(C_1 + C_2) + \lambda) B_C^{-1} G_C(\lambda) \zeta_\psi \\ & \quad + \psi_0 - (1 + \bar{C}_2 \bar{B}_C^{-1} (-(C_1 + C_2) + \lambda) B_C^{-1}) G_C(\lambda) \zeta_\psi \\ & \quad - \lambda \bar{B}_C^{-1} (-(C_1 + C_2) + \lambda) B_C^{-1} G_C(\lambda) \zeta_\psi \\ &= \hat{C}_2 \phi_0 + \psi_0 - G_C \zeta_\psi \end{aligned}$$

and

$$\begin{aligned} & C_1 \psi_\lambda + \bar{A} \phi_\lambda + \lambda \bar{C}_2 \bar{B}_C^{-1} (-(C_1 + C_2) + \lambda) B_C^{-1} G_C(\lambda) \zeta_\psi \\ &= \hat{C}_1 \psi_0 - \hat{C}_1 (1 - \bar{C}_2 \bar{B}_C^{-1} (C_1 + C_2) B_C^{-1}) G_C(\lambda) \zeta_\psi \\ & \quad + \hat{A} \phi_0 - \hat{A} \bar{B}_C^{-1} (C_1 + C_2) B_C^{-1} G_C(\lambda) \zeta_\psi \\ & \quad + \hat{C}_2 G_C \zeta_\psi - \hat{C}_2 G_C(\lambda) \zeta_\psi \\ &= \hat{C}_1 \psi_0 - \hat{C}_1 G_C(\lambda) \zeta_\psi - \hat{C}_2 G_C(\lambda) \zeta_\psi + \hat{A} \phi_0 + \hat{C}_2 G_C \zeta_\psi \\ & \quad (-\hat{A} + \hat{C}_1 \bar{C}_2) \bar{B}_C^{-1} (C_1 + C_2) B_C^{-1} G_C(\lambda) \zeta_\psi \\ &= \hat{C}_1 \psi_0 + \hat{A} \phi_0 + \hat{C}_2 G_C \zeta_\psi. \end{aligned}$$

□

By Lemma 4.8 we can define the Hilbert space $(\mathcal{H}_\Theta, \langle \langle \cdot, \cdot \rangle \rangle_{\mathcal{H}_\Theta})$ by

$$\begin{aligned} \mathcal{H}_\Theta &:= \{(\phi, \psi) \in \mathcal{V} \times \mathcal{H}_{-1} : \phi = \phi_0 + G_C \zeta_\phi, \\ & \quad \psi = \psi_0 - \hat{C}_2 G_C \zeta_\phi, (\phi_0, \psi_0, \zeta_\phi) \in \mathcal{H}_C \oplus \mathfrak{h}\} \end{aligned}$$

with scalar product

$$\langle \langle (\phi, \psi), (\tilde{\phi}, \tilde{\psi}) \rangle \rangle_{\mathcal{H}_\Theta} := \langle \langle (\phi_0, \psi_0, \zeta_\phi), (\tilde{\phi}_0, \tilde{\psi}_0, \zeta_{\tilde{\phi}}) \rangle \rangle_{C, \Theta},$$

so that map

$$U : \mathcal{H}_C \oplus \mathfrak{h}_\Theta \rightarrow \mathcal{H}_\Theta, \quad U(\phi_0, \psi_0, \zeta_\phi) := (\phi_0 + G_C \zeta_\phi, \psi_0 - \hat{C}_2 G_C \zeta_\phi)$$

is unitary. Thus in conclusion we have the following

Theorem 4.11. *Suppose H3.0, H3.1, H3.2, H4.1.1, H4.2.1 and H4.3.1 hold true. Then the linear operator*

$$W_\Theta : D(W_\Theta) \subseteq \mathcal{H}_\Theta \rightarrow \mathcal{H}_\Theta,$$

$$\begin{aligned}
D(W_\Theta) &:= \{(\phi, \psi) \in \mathcal{H}_\Theta : \phi_0 = \phi_\lambda + \bar{B}_C^{-1}(C_1 + C_2)B_C^{-1}G_C(\lambda)\zeta_\psi, \\
&\quad \psi_0 = \psi_\lambda + (1 - \bar{C}_2\bar{B}_C^{-1}(C_1 + C_2)B_C^{-1})G_C(\lambda)\zeta_\psi, \\
&\quad \phi_\lambda \in \bar{\mathcal{H}}_2, \psi_\lambda \in \mathcal{H}_1, \zeta_\psi \in \mathfrak{h}, \Theta\zeta_\phi = \bar{\tau}\phi_0\}.
\end{aligned}$$

$$W_\Theta(\phi, \psi) := U\tilde{W}_\Theta U^*(\phi, \psi) = (\hat{C}_2\phi_0 + \psi_0, \hat{C}_1\psi_0 + \hat{A}\phi_0),$$

is skew-adjoint. It coincides with

$$\begin{aligned}
W_g : \bar{\mathcal{H}}_2 \times \mathcal{H}_1 &\subseteq \mathcal{H}_C \rightarrow \mathcal{H}_C \\
W_g(\phi, \psi) &:= (\bar{C}_2\phi + \psi, C_1\psi + \bar{A}\phi)
\end{aligned}$$

on the dense set

$$D(W_g) \cap D(W_\Theta) = \{\phi \in \bar{\mathcal{H}}_2 : \tau\phi = 0\} \times \mathcal{H}_1.$$

5. EXAMPLES

– *Example 1.* Let A_0 be the negative and injective self-adjoint operator on $\mathcal{H}_0 = L^2(0, \infty)$ corresponding to the second derivative operator with Dirichlet boundary conditions at zero, i.e.

$$A_0 : D(A_0) \subset L^2(0, \infty) \rightarrow L^2(0, \infty), \quad A_0\phi := \phi'',$$

where $\mathcal{H}_2 \equiv D(A_0) \equiv H_0^2(0, \infty)$,

$$H_0^2(0, \infty) := \{\phi \in L^2(0, \infty) : \phi'' \in L^2(0, \infty), \phi(0_+) = 0\}.$$

Let B_0 be the positive and injective self-adjoint operator defined by $B_0 := \sqrt{-A_0}$. We have $\mathcal{H}_1 \equiv D(B_0) \equiv H_0^1(0, \infty)$, where

$$H_0^1(0, \infty) := \{\phi \in L^2(0, \infty) : \phi' \in L^2(0, \infty), \phi(0_+) = 0\},$$

with scalar product

$$\langle \phi_1, \phi_2 \rangle_1 := \langle \phi_1, \phi_2 \rangle + \langle \phi_1', \phi_2' \rangle.$$

Here $\langle \cdot, \cdot \rangle$ denotes here the usual scalar product on $L^2(0, \infty)$.

Let us now consider $\bar{\mathcal{H}}_1 \equiv \bar{H}_0^1(0, \infty)$, the completion of $H_0^1(0, \infty)$ with respect to the scalar product

$$[\phi_1, \phi_2]_1 := \langle \phi_1', \phi_2' \rangle.$$

One has

$$\bar{H}_0^1(0, \infty) := \left\{ \phi \in \bigcup_{b>0} L^2(0, b) : \phi' \in L^2(0, \infty), \phi(0_+) = 0 \right\},$$

and then

$$\bar{H}_0^2(0, \infty) := \left\{ \phi \in \bigcup_{b>0} L^2(0, b) : \phi', \phi'' \in L^2(0, \infty), \phi(0_+) = 0 \right\}.$$

Moreover \bar{A}_0 acts on $\bar{H}_0^2(0, \infty)$ as the second (distributional) derivative operator. The resolvent $(-A_0 + \lambda^2)^{-1}$ has an integral kernel given by

$$\mathcal{G}_D(\lambda; x, y) = \frac{e^{-|\lambda||x-y|} - e^{-|\lambda|(x+y)}}{2|\lambda|}.$$

We consider now the negative and injective self-adjoint operator on $\bigoplus_{k=1}^n L^2(0, \infty)$ defined by $A := \bigoplus_{k=1}^n A_0$ and the bounded linear map

$$\tau : \bigoplus_{k=1}^n \bar{H}_0^2(0, \infty) \rightarrow \mathbb{C}^n, \quad \tau(\phi_1, \dots, \phi_n) := (\phi_1'(0_+), \dots, \phi_n'(0_+)).$$

Obviously $\tau_\Theta(\phi_1, \dots, \phi_n, \zeta) := \tau(\phi_1, \dots, \phi_n) - \Theta\zeta$ satisfies hypothesis H.3.0 for any positive and injective Hermitean Θ .

One has that $G(\lambda) : \mathbb{C}^n \rightarrow \bigoplus_{k=1}^n L^2(0, \infty)$ is represented by the vector in $\bigoplus_{k=1}^n L^2(0, \infty)$ given by

$$\mathcal{G}_\lambda(x_1, \dots, x_n) := (e^{-|\lambda|x_1}, \dots, e^{-|\lambda|x_n}),$$

while $\bar{G}(\lambda) : \mathbb{C}^n \rightarrow \bigoplus_{k=1}^n \bar{H}_0^2(0, \infty)$ is represented by the vector in $\bigoplus_{k=1}^n \bar{H}_0^2(0, \infty)$ given by

$$\begin{aligned} & \bar{\mathcal{G}}_\lambda(x_1, \dots, x_n) \\ & \equiv \lim_{\epsilon \rightarrow 0} \left(\int_0^\infty dx_1 \mathcal{G}_D(\epsilon; x_1, y_1) \mathcal{G}_\lambda(y_1), \dots, \int_0^\infty dx_n \mathcal{G}_D(\epsilon; x_n, y_n) \mathcal{G}_\lambda(y_n) \right) \\ & = \left(\frac{e^{-|\lambda|x_1} - 1}{|\lambda|^2}, \dots, \frac{1 - e^{-|\lambda|x_1}}{|\lambda|^2} \right). \end{aligned}$$

Therefore

$$\Gamma_\Theta(\lambda) = -\lambda\tau\bar{G}(\lambda) - \frac{1}{\lambda}\Theta = -\frac{1}{\lambda}(|\lambda| + \Theta).$$

Note that, since $\mathcal{G}_\lambda(0_+) \neq 0$, $\text{Ran}(G(\lambda)) \cap H_0^1(0, \infty) = \{0\}$ and H.3.1 is satisfied.

Hypothesis H.3.2 is satisfied by taking $\mathcal{V} = \bigcup_{b>0} L^2((0, b)^n)$ and $G : \mathbb{C} \rightarrow \bigcup_{b>0} L^2((0, b)^n)$ is represented by the constant vector

$$\mathcal{G}(x_1, \dots, x_n) = (\mathcal{G}_\lambda + \lambda^2 \bar{\mathcal{G}}_\lambda)(x_1, \dots, x_n) = (1, \dots, 1).$$

Defining

$$\bar{H}^1(0, \infty) := \left\{ \phi \in \bigcup_{b>0} L^2(0, b) : \phi' \in L^2(0, \infty) \right\},$$

$$H^1(0, \infty) := \bar{H}^1(0, \infty) \cap L^2(0, \infty),$$

and

$$\bar{H}^2(0, \infty) := \left\{ \phi \in \bigcup_{b>0} L^2(0, b) : \phi', \phi'' \in L^2(0, \infty) \right\},$$

$$H^2(0, \infty) := \bar{H}^2(0, \infty) \cap L^2(0, \infty),$$

one has

$$\begin{aligned} \bar{K}^1 &:= \left\{ \Phi = \Phi_0 + \zeta_\Phi \mathcal{G}, \Phi_0 \in \bigoplus_{k=1}^n \bar{H}_0^1(0, \infty), \zeta_\Phi \in \mathbb{C}^n \right\} \\ &\equiv \bigoplus_{k=1}^n \bar{H}^1(0, \infty), \end{aligned}$$

$$\begin{aligned} \bar{K}^2 &:= \left\{ \Phi = \Phi_0 + \zeta_\Phi \mathcal{G}, \Phi_0 \in \bigoplus_{k=1}^n \bar{H}_0^2(0, \infty), \zeta_\Phi \in \mathbb{C}^n \right\} \\ &\equiv \bigoplus_{k=1}^n \bar{H}^2(0, \infty), \end{aligned}$$

and

$$K^1 := \bar{K}^1 \cap \bigoplus_{k=1}^n L^2(0, \infty) \equiv \bigoplus_{k=1}^n H^1(0, \infty).$$

$$K^2 := \bar{K}^2 \cap \bigoplus_{k=1}^n L^2(0, \infty) \equiv \bigoplus_{k=1}^n H^2(0, \infty).$$

One makes $\bigoplus_{k=1}^n (\bar{H}^1(0, \infty) \oplus L^2(0, \infty))$ a Hilbert space by the scalar product

$$\begin{aligned} &\langle\langle (\Phi, \Psi), (\tilde{\Phi}, \tilde{\Psi}) \rangle\rangle \\ &:= \sum_{1 \leq k \leq n} \langle \phi'_k, \tilde{\phi}'_k \rangle + \sum_{1 \leq k \leq n} \langle \psi_k, \tilde{\psi}_k \rangle + \sum_{1 \leq k, j \leq n} \Theta_{kj} \bar{\phi}_k(0_+) \tilde{\phi}_j(0_+). \end{aligned}$$

Here we put $\Phi \equiv (\phi_1, \dots, \phi_n)$, $\Psi \equiv (\psi_1, \dots, \psi_n)$ and we used the fact that $\zeta_\Phi = (\phi_1(0_+), \dots, \phi_n(0_+))$.

By Theorem 3.6 we define now skew-adjoint operators W_Θ corresponding to wave equations on star-like graphs: the operator

$$W_\Theta : D(W_\Theta) \rightarrow \bigoplus_{k=1}^n (\bar{H}^1(0, \infty) \oplus L^2(0, \infty)),$$

$$D(W_\Theta) := \left\{ \Phi \in \bigoplus_{k=1}^n \bar{H}^2(0, \infty) : \phi'_k(0_+) + \sum_{1 \leq j \leq n} \Theta_{k,j} \phi_j(0_+) = 0, \ 1 \leq k \leq n \right\} \\ \oplus \bigoplus_{k=1}^n H^1(0, \infty),$$

$$W_\Theta(\Phi, \Psi) := (\Psi, \bar{A}\Phi_0) \equiv \left(\Psi, \bigoplus_{k=1}^n \bar{A}_0 \Phi_0 \right) \equiv (\psi_1, \dots, \psi_n, \phi''_1, \dots, \phi''_n)$$

is skew-adjoint and coincides with

$$W : \bigoplus_{k=1}^n (\bar{H}_0^2(0, \infty) \oplus H_0^1(0, \infty)) \rightarrow \bigoplus_{k=1}^n (\bar{H}_0^1(0, \infty) \oplus L^2(0, \infty)),$$

$$W(\Phi, \Psi) := (\Psi, \bar{A}\Phi) \equiv \left(\Psi, \bigoplus_{k=1}^n \bar{A}_0 \Phi \right) \equiv (\psi_1, \dots, \psi_n, \phi''_1, \dots, \phi''_n)$$

on the set

$$\left\{ \Phi \in \bigoplus_{k=1}^n \bar{H}_0^2(0, \infty) : \phi'_k(0_+) = 0, \ 1 \leq k \leq n \right\} \oplus \bigoplus_{k=1}^n H_0^1(0, \infty).$$

Moreover, by Theorem 3.7, the linear operator

$$D(A_\Theta) := \left\{ \Phi \in \bigoplus_{k=1}^n H^2(0, \infty) : \phi'_k(0_+) + \sum_{1 \leq j \leq n} \Theta_{k,j} \phi_j(0_+) = 0, \ 1 \leq k \leq n \right\},$$

$$A_\Theta : D(A_\Theta) \subset \bigoplus_{k=1}^n L^2(0, \infty) \rightarrow \bigoplus_{k=1}^n L^2(0, \infty), \quad A_\Theta \Phi := (\phi''_1, \dots, \phi''_n),$$

is negative, injective self-adjoint, its resolvent has an integral kernel given by

$$(-A_\Theta + \lambda^2)^{-1}(x_1, \dots, x_n, y_1, \dots, y_n) \\ = \mathcal{G}_D(\lambda; x_1, y_1) \cdots \mathcal{G}_D(\lambda; x_n, y_n) + \sum_{1 \leq k, j \leq n} (\Theta + |\lambda|)_{kj}^{-1} e^{-|\lambda|(x_k + y_j)}.$$

The operator A_Θ is of the class of Laplacian operators on star-like graphs (see e.g. [9] and references therein) and the positive quadratic form corresponding to $-A_\Theta$ is

$$\mathcal{Q}_\Theta : \bigoplus_{k=1}^n H^1(0, \infty) \subset \bigoplus_{k=1}^n L^2(0, \infty) \rightarrow \mathbb{R},$$

$$\mathcal{Q}_\Theta(\Phi) := \sum_{1 \leq k \leq n} \|\phi'_k\|_2^2 + \sum_{1 \leq k, j \leq n} \Theta_{kj} \bar{\phi}_k(0_+) \phi_j(0_+).$$

Let B be the injective selfadjoint operator on L^2 , the Hilbert space of square integrable functions on \mathbb{R}^3 , given by $B = \sqrt{-\Delta}$. Then \mathcal{H}_1 coincides with the Sobolev space H^1 of L^2 function with L^2 distributional derivatives. $\bar{\mathcal{H}}_1$ is nothing else than the usual Riesz potential space \bar{H}^1 given by the set of tempered distributions with a Fourier transform (denoted by F) which is square integrable w.r.t. the measure with density $|k|^2$. The operator \bar{B} is then defined by

$$F\bar{B}\phi(k) := |k| F\phi(k).$$

The space $\bar{\mathcal{H}}_2$ coincides with the space \bar{H}^2 of distributions in \bar{H}^1 with a Fourier transform which is square integrable w.r.t. the measure with density $|k|^2(|k|^2 + 1)$. By Sobolev embedding theorems the elements of both \bar{H}^1 and \bar{H}^2 are ordinary functions. Indeed

$$\bar{H}^2 \subset \bar{H}^1 \subset L^6(\mathbb{R}^3), \quad \bar{H}^2 \subset C_b(\mathbb{R}^3),$$

the embeddings being continuous. The linear operator $\bar{A} := -B\bar{B}$ acts on \bar{H}^2 as the distributional Laplacean Δ , or equivalently

$$FA\phi(k) := -|k|^2 F\phi(k).$$

In the sequel $\langle \cdot, \cdot \rangle$ will denote the scalar product on L^2 . More generally, for any ϕ, φ such that $\phi\varphi$ is integrable, we will use the notation

$$\langle \phi, \varphi \rangle := \int_{\mathbb{R}^3} dx \bar{\phi}(x) \varphi(x).$$

Moreover $*$ will denote convolution.

– *Example 2.* On the Hilbert space $\bar{H}^1 \oplus L^2$ with scalar product

$$\langle (\phi_1, \psi_1), (\phi_2, \psi_2) \rangle := \langle \nabla \phi_1, \nabla \phi_2 \rangle + \langle \psi_1, \psi_2 \rangle$$

we consider the skew-adjoint operator

$$W : \bar{H}^2 \oplus H^1 \subset \bar{H}^1 \oplus L^2 \rightarrow \bar{H}^1 \oplus L^2, \quad W(\phi, \psi) := (\psi, \Delta \phi).$$

by Theorem 2.5 its resolvent is given by

$$(W + \lambda)^{-1}(\phi, \psi) = (\mathcal{G}_\lambda * (\psi + \lambda\phi), -\phi + \lambda\mathcal{G}_\lambda * (\psi + \lambda\phi)),$$

where

$$\mathcal{G}_\lambda(x) = \frac{e^{-|\lambda x|}}{4\pi|x|}, \quad \mathcal{G} \equiv \mathcal{G}_0.$$

Given an injective and positive Hermitean $n \times n$ matrix $\Theta = (\theta_{ij})$, we consider the Hilbert space $\bar{H}^1 \oplus L^2 \oplus \mathbb{C}^n$ with scalar product

$$\langle (\phi_1, \psi_1, \zeta_1), (\phi_2, \psi_2, \zeta_2) \rangle_\Theta := \langle \nabla \phi_1, \nabla \phi_2 \rangle + \langle \psi_1, \psi_2 \rangle + (\Theta \zeta_1, \zeta_2)$$

where (\cdot, \cdot) denotes the scalar product on \mathbb{C}^n .

Given $Y = \{y_1, \dots, y_n\} \subset \mathbb{R}^3$, let

$$\tau : \bar{H}^2 \rightarrow \mathbb{C}^n, \quad (\tau\phi_0)_i := \phi_0(y_i), \quad 1 \leq i \leq n.$$

Such a map satisfies H3.1 since $\tau^*\zeta = \zeta_i \delta_{y_i}$, where δ_y denotes Dirac's mass at y . Here and below we use Einstein's summation convention: repeated indices mean summation.

We define then the continuous linear map, which obviously satisfies H3.0,

$$\tau_\Theta : \bar{H}^2 \oplus L^2 \oplus \mathbb{C}^n \rightarrow \mathbb{C}^n, \quad \tau_\Theta(\phi_0, \psi, \zeta)_i := \phi_0(y_i) - \theta_{ij} \zeta_j.$$

Thus, according to the definitions (3.2) and (3.3) one obtains

$$\check{G}_\Theta(\lambda)(\phi, \psi, \zeta)_i = \langle \mathcal{G}_\lambda^i, \psi + \lambda\phi \rangle - \frac{1}{\lambda} \theta_{ij} \zeta_j,$$

where $\mathcal{G}_\lambda^i(x) := \mathcal{G}_\lambda(x - y_i)$, and

$$G_\Theta(\lambda)\zeta = \left(\lambda \zeta_i \mathcal{G} * \mathcal{G}_\lambda^i, -\zeta_i \mathcal{G}_\lambda^i, -\frac{1}{\lambda} \zeta \right).$$

Therefore, putting $\mathcal{G}^i := \mathcal{G}_0^i$, by (3.4),

$$\begin{aligned} (\Gamma_\Theta(\lambda)\zeta)_i &= -(\tau_\Theta G_\Theta(\lambda)\zeta)_i \\ &= -\left(\lambda \langle \mathcal{G}^i, \mathcal{G}_\lambda^j \rangle + \frac{1}{\lambda} \theta_{ij} \right) \zeta_j \\ &= -\frac{1}{\lambda} \left(\frac{|\lambda|}{4\pi} \zeta_i + \left((1 - \delta_{ij}) \frac{1 - e^{-|\lambda(y_i - y_j)|}}{4\pi|y_i - y_j|} + \theta_{ij} \right) \zeta_j \right), \end{aligned}$$

i.e., defining

$$\begin{aligned} \Theta_Y &:= \left((1 - \delta_{ij}) \frac{1}{4\pi|y_i - y_j|} \right), \quad M(\lambda) := \left((1 - \delta_{ij}) \frac{e^{-|\lambda(y_i - y_j)|}}{4\pi|y_i - y_j|} \right), \\ \Gamma_\Theta(\lambda) &= -\frac{1}{\lambda} \left(\Theta + \Theta_Y + \frac{|\lambda|}{4\pi} - M(\lambda) \right). \end{aligned}$$

Since H3.2 is verified by taking $\mathcal{V} = L_{\text{loc}}^2$, we put, defining $\mathcal{G}^i(x) := \mathcal{G}(x - y_i)$,

$$\begin{aligned} \bar{K}^1 &:= \{ \phi \in L_{\text{loc}}^2 : \phi = \phi_0 + \zeta_\phi^i \mathcal{G}^i, \phi_0 \in \bar{H}^1, \zeta_\phi \in \mathbb{C}^n \}, \\ \bar{K}^2 &:= \{ \phi \in L_{\text{loc}}^2 : \phi = \phi_0 + \zeta_\phi^i \mathcal{G}^i, \phi_0 \in \bar{H}^2, \zeta_\phi \in \mathbb{C}^n \}, \\ K^1 &:= \bar{K}^1 \cap L^2, \end{aligned}$$

and making $\bar{K}^1 \oplus L^2$ a Hilbert space by the scalar product

$$\begin{aligned} &\langle \langle (\phi, \psi), (\tilde{\phi}, \tilde{\psi}) \rangle \rangle_{\bar{K}^1 \oplus L^2} \\ &:= \langle \nabla \phi_0, \nabla \tilde{\phi}_0 \rangle + \langle \psi, \tilde{\psi} \rangle + (\Theta \zeta_\phi, \zeta_{\tilde{\phi}}), \end{aligned}$$

by Theorem 3.6 the operator

$$\begin{aligned} D(W_\Theta) &:= \{ \phi \in \bar{K}^2 : \theta_{ij} \zeta_\phi^j = \phi_0(y_i) \} \oplus K^1, \\ W_\Theta : D(W_\Theta) &\subset \bar{K}^1 \oplus L^2 \rightarrow \bar{K}^1 \oplus L^2, \\ W_\Theta(\phi, \psi) &:= (\psi, \Delta \phi_0) \equiv (\psi, \Delta \phi + \zeta_\phi^j \delta_{y_j}) \end{aligned}$$

is skew-adjoint and coincides with W on the set

$$\{ \phi \in \bar{H}^2 : \phi(y) = 0, y \in Y \} \oplus H^1.$$

In the case $Y = \{0\}$ this operator coincides with the one constructed in [4]. By Theorem 3.7, the positive quadratic form

$$\mathcal{Q}_\Theta : K_1 \rightarrow \mathbb{R}, \quad \mathcal{Q}_\Theta(\phi) := \|\nabla \phi_0\|_{L^2}^2 + \|\Theta^{1/2} \zeta_\phi\|_{\mathbb{C}^n}^2$$

is closed and the corresponding self-adjoint operator $-\Delta_\Theta$ is defined by

$$\begin{aligned} D(\Delta_\Theta) &= \{ \phi \in \bar{K}^2 \cap L^2 : \theta_{ij} \zeta_\phi^j = \phi_0(y_i) \}, \\ \Delta_\Theta \phi &:= \Delta \phi_0. \end{aligned}$$

It coincides with Δ on the set $\{ \phi \in H^2 : \phi(y) = 0, y \in Y \}$. Its resolvent is given by

$$(-\Delta_\Theta + \lambda^2)^{-1} \psi = \mathcal{G}_\lambda * \psi + \left(\Theta + \Theta_Y + \frac{|\lambda|}{4\pi} - M(\lambda) \right)_{ij}^{-1} \langle \mathcal{G}_\lambda^i, \psi \rangle \mathcal{G}_\lambda^j.$$

This operator is of the class of point perturbation of the Laplacian (see [1] and references therein).

– *Example 3.* Given $v \in \mathbb{R}^3$, $|v| < 1$, we consider the skew-adjoint operator

$$W^v : \bar{H}^2 \times H^1 \subseteq H^v \rightarrow H^v,$$

$$W^v(\phi, \psi, z) := (L_v \phi + \psi, L_v \psi + \Delta \phi),$$

where $L_v := v \cdot \nabla$ and $H^v = \bar{H}^1 \times L^2$ with scalar product

$$\langle \langle (\phi_1, \psi_1), (\phi_2, \psi_2) \rangle \rangle_v := \langle \nabla \phi_1, \nabla \phi_2 \rangle + \langle L_v \phi_1, \psi_2 \rangle + \langle \psi_1, L_v \phi_2 \rangle + \langle \psi_1, \psi_2 \rangle.$$

Hypotheses H4.1.1 and H4.1.2 are satisfied with $C_1 = C_2 = L_v$ and, by Theorem 4.2, with $C = 2L_v$ and $B = (-\Delta + L_v^2)$, the resolvent of W^v is given by

$$\begin{aligned} (W^v + \lambda)^{-1}(\phi, \psi) &= (\mathcal{G}_\lambda^v * (\psi + (-L_v + \lambda)\phi), \\ &\quad -\phi + (-L_v + \lambda) \mathcal{G}_\lambda^v * (\psi + (-L_v + \lambda)\phi)), \end{aligned}$$

where

$$F\mathcal{G}_\lambda^v(k) = \frac{1}{(2\pi)^{3/2}} \frac{1}{|k|^2 + (iv \cdot k + \lambda)^2}.$$

Let

$$\tau : \bar{H}^2 \rightarrow \mathbb{C}, \quad \tau \phi_0 := \phi_0(0).$$

By Example 2 we know that such a map satisfies H3.1. For any real $\theta > 0$, define now the linear map, which obviously satisfies H3.0,

$$\bar{\tau}_\theta : D(\bar{\tau}) \times L^2 \times \mathbb{C} \subseteq H^v \oplus \mathbb{C} \rightarrow \mathbb{C}, \quad \bar{\tau}_\theta(\phi, \psi, \zeta) := \bar{\tau}\phi - \theta\zeta,$$

where, denoting by $\langle \phi \rangle_R$ the average of ϕ over the sphere of radius R ,

$$D(\bar{\tau}) := \left\{ \phi \in \bar{H}^1 : \lim_{R \downarrow 0} \langle \phi \rangle_R \text{ exists and is finite} \right\}, \quad \bar{\tau}\phi := \lim_{R \downarrow 0} \langle \phi \rangle_R.$$

Since, by Fourier transform,

$$\bar{\tau}\phi = \lim_{R \downarrow 0} \frac{1}{(2\pi)^{3/2}} \int_{\mathbb{R}^3} dk \frac{\sin R|k|}{R|k|} F\phi(k),$$

with reference to the notations of Section 4, we are taking here the regularizing family

$$J_\nu = \frac{1}{(2\pi)^{3/2}} \left(\nu \sqrt{-\Delta} \right)^{-1} \sin \nu \sqrt{-\Delta}.$$

Thus $\bar{H}^2 \subset D(\bar{\tau})$ and $\bar{\tau}\phi = \phi(0)$ for any $\phi \in \bar{H}^2$. Then one obtains, by (4.2) and (4.4),

$$\check{G}_\theta^v(\lambda)(\phi, \psi, \zeta) = \langle \mathcal{G}_\lambda^v, \psi + (-L_v + \lambda)\phi \rangle - \theta\lambda^{-1}\zeta$$

and

$$G_\theta^v(\lambda)\zeta = \zeta((-L_v + \lambda)\mathcal{G}^v * \mathcal{G}_\lambda^v, -\mathcal{G}_\lambda^v, -\lambda^{-1}),$$

where

$$\mathcal{G}^v(x) := \mathcal{G}_0^v(x) \equiv \frac{1}{4\pi\sqrt{|x|^2 - |v \wedge x|^2}}.$$

Regarding hypothesis H4.3.1 one has

$$\begin{aligned} & \bar{\tau}L_v\mathcal{G}^v * \mathcal{G}_\lambda^v \\ &= \frac{1}{(2\pi)^3} \lim_{R \downarrow 0} \int_{\mathbb{R}^3} dk \frac{\sin R|k|}{R|k|} \frac{1}{|k|^2 - (v \cdot k)^2} \frac{-iv \cdot k}{|k|^2 + (iv \cdot k + \lambda)^2} \\ &= \frac{1}{(2\pi)^2} \lim_{R \downarrow 0} \int_0^\infty dr \frac{\sin Rr}{Rr} \int_0^\pi \frac{d\theta \sin \theta}{1 - (|v| \cos \theta)^2} \frac{-i|v|r \cos \theta}{r^2 + (i|v|r \cos \theta + \lambda)^2} \\ &= \frac{1}{(2\pi)^2|v|} \lim_{R \downarrow 0} \int_0^\infty dr \frac{\sin Rr}{Rr} \int_{-|v|}^{|v|} \frac{ds}{1 - s^2} \frac{irs}{r^2 + (-irs + \lambda)^2} \\ &= \frac{1}{(2\pi)^2|v|} \lim_{R \downarrow 0} \int_0^\infty dr \frac{\sin Rr}{Rr} \int_{-|v|}^{|v|} \frac{ds}{1 - s^2} \frac{-2\lambda r^2 s^2}{((1 - s^2)r^2 + \lambda^2)^2 + 4\lambda^2 r^2 s^2} \\ &= -\frac{4\lambda}{(2\pi)^2|v|} \int_0^\infty dr \int_0^{|v|} ds \frac{s^2}{1 - s^2} \frac{r^2}{((1 - s^2)r^2 + \lambda^2)^2 + 4\lambda^2 r^2 s^2}, \end{aligned}$$

and

$$\begin{aligned}
& \tau \mathcal{G}^v * \mathcal{G}_\lambda^v \\
&= \frac{1}{(2\pi)^3} \int_{\mathbb{R}^3} dk \frac{1}{|k|^2 - (v \cdot k)^2} \frac{1}{|k|^2 + (iv \cdot k + \lambda)^2} \\
&= \frac{1}{(2\pi)^2} \int_0^\infty dr \int_0^\pi \frac{d\theta \sin \theta}{1 - (|v| \cos \theta)^2} \frac{1}{r^2 + (i|v|r \cos \theta + \lambda)^2} \\
&= \frac{1}{(2\pi)^2 |v|} \int_0^\infty dr \int_{-|v|}^{|v|} \frac{ds}{1 - s^2} \frac{1}{r^2 + (-irs + \lambda)^2} \\
&= \frac{2}{(2\pi)^2 |v|} \int_0^\infty dr \int_0^{|v|} \frac{ds}{1 - s^2} \frac{(1 - s^2)r^2 + \lambda^2}{((1 - s^2)r^2 + \lambda^2)^2 + 4\lambda^2 r^2 s^2}.
\end{aligned}$$

Thus H4.3.1 holds true and

$$\begin{aligned}
& \Gamma_\theta^v(\lambda) := -\bar{\tau}_\theta G_\theta^v(\lambda) \\
&= - \left(\frac{\theta}{\lambda} + \frac{2\lambda}{(2\pi)^2 |v|} \int_0^\infty dr \int_0^{|v|} \frac{ds}{1 - s^2} \frac{(1 + s^2)r^2 + \lambda^2}{((1 - s^2)r^2 + \lambda^2)^2 + 4\lambda^2 r^2 s^2} \right) \\
&= - \left(\frac{\theta}{\lambda} + \frac{\lambda}{(2\pi)^2 |v|} \int_0^{|v|} \frac{ds}{1 - s^2} \int_{-\infty}^\infty dr \frac{(1 + s^2)r^2 + \lambda^2}{((1 - s^2)r^2 + \lambda^2)^2 + 4\lambda^2 r^2 s^2} \right) \\
&= - \frac{1}{\lambda} \left(\theta + \frac{|\lambda|}{8\pi |v|} \int_0^{|v|} ds \frac{2 - s^2}{(1 - s^2)^2} \right) \\
&= - \frac{1}{\lambda} \left(\theta + \frac{|\lambda|}{16\pi} \left(\frac{1}{1 - |v|^2} + \frac{3}{2} \frac{1}{|v|} \ln \frac{1 + |v|}{1 - |v|} \right) \right).
\end{aligned}$$

Note that

$$\lim_{|v| \downarrow 0} \Gamma_\theta^v(\lambda) = -\frac{1}{\lambda} \left(\theta + \frac{|\lambda|}{4\pi} \right),$$

in accordance with the previous example when $Y = \{0\}$.

Since H3.2 is verified by taking $\mathcal{V} = L_{\text{loc}}^2$, defining the Hilbert space

$$\begin{aligned}
H_\theta^v &:= \{(\phi, \psi) \in L_{\text{loc}}^2 \times H^{-1} : \phi = \phi_0 + \zeta_\phi \mathcal{G}^v, \\
&\psi = \psi_0 - \zeta_\phi L_v \mathcal{G}^v, (\phi_0, \psi_0, \zeta_\phi) \in \bar{H}^v \oplus \mathbb{C}\}
\end{aligned}$$

with scalar product

$$\begin{aligned}
\langle (\phi, \psi), (\tilde{\phi}, \tilde{\psi}) \rangle_{H_\theta^v} &:= \langle \nabla \phi_0, \nabla \tilde{\phi}_0 \rangle + \langle L_v \tilde{\phi}_0, \tilde{\psi}_0 \rangle \\
&+ \langle \psi_0, L_v \tilde{\phi}_0 \rangle + \langle \psi_0, \tilde{\psi}_0 \rangle + \theta \zeta_\phi^* \zeta_{\tilde{\phi}},
\end{aligned}$$

by Theorem 4.11 the operator

$$W_\theta^v : D(W_\theta^v) \subseteq H_\theta^v \rightarrow H_\theta^v,$$

$$\begin{aligned}
D(W_\theta^v) &:= \{(\phi, \psi) \in H_\theta^v : \phi_0 = \phi_\lambda + 2\zeta_\psi L_v \mathcal{G}^v * \mathcal{G}_\lambda^v, \\
&\quad \psi_0 = \psi_\lambda + \zeta_\psi (\mathcal{G}_\lambda^v - 2L_v^2 \mathcal{G}^v * \mathcal{G}_\lambda^v), \\
&\quad \phi_\lambda \in \bar{H}^2, \psi_\lambda \in H^1, \zeta_\psi \in \mathbb{C}, \theta \zeta_\phi = \bar{\tau} \phi_0\}.
\end{aligned}$$

$$\begin{aligned}
W_\theta^v(\phi, \psi) &:= (L_v \phi_0 + \psi_0, L_v \psi_0 + \Delta \phi_0) \\
&\equiv (L_v \phi + \psi, L_v \psi + \Delta \phi + \zeta_\phi \delta_0),
\end{aligned}$$

is skew-adjoint. It coincides with W^v on the dense set

$$\{\phi \in \bar{H}^2 : \phi(0) = 0\} \times H^1.$$

– *A digression on the classical electrodynamics of a point particle.* Let us begin with a discussion at the euristic level ignoring the singular behaviour due to the self-energy of the point particle.

In the Coulomb gauge the Maxwell-Lorentz system, i.e the non-linear infinite dimensional dynamical system describing a (relativistic) charged point particle interacting with the self-generated radiation field, is given by the equations

$$\begin{aligned}
\dot{\mathbf{A}} &= \mathbf{E} \\
\dot{\mathbf{E}} &= \Delta \mathbf{A} + 4\pi e M v \delta_q \\
\dot{q} &= v \\
\dot{p} &= e \nabla \mathbf{A}(q) \cdot v,
\end{aligned}$$

where

$$v = v(\mathbf{A}, q, p) := \frac{p - e \mathbf{A}(q)}{\sqrt{|p - e \mathbf{A}(q)|^2 + m^2}}$$

or, equivalently,

$$p = p(\mathbf{A}, q, v) = \frac{mv}{\sqrt{1 - |v|^2}} + e \mathbf{A}(q).$$

Here we put $c = 1$, where c denotes the velocity of light, e denotes the electric charge, M is the projection onto the divergenceless vector fields, $\mathbf{A} \equiv (A_1, A_2, A_3)$, $\operatorname{div} \mathbf{A} = 0$, is the vector potential of the electromagnetic field, q , v , $|v| < 1$, and p denote the particle position, velocity and momentum respectively. Since the total (particle + field) momentum

$$\Pi := p - \frac{1}{4\pi} \langle \mathbf{E}, \nabla \mathbf{A} \rangle, \quad \langle \mathbf{E}, \nabla \mathbf{A} \rangle := \sum_{j=1}^3 \int_{\mathbb{R}^3} dx E_j(x) \nabla A_j(x),$$

is conserved, the above dynamical system can be reduced. Indeed, by defining the fields

$$\Phi(x) := A(x + q), \quad \Psi(x) := E(x + q),$$

the Maxwell-Lorentz system can be re-written as

$$\begin{aligned} \dot{\Phi} &= v \cdot \nabla \Phi + \Psi \\ \dot{\Psi} &= v \cdot \nabla \Psi + \Delta \Phi + 4\pi e M v \delta_0 \\ \dot{q} &= v \\ \dot{\Pi} &= 0, \end{aligned}$$

where now

$$v = v(\Phi, \Psi) := \frac{p - e \Phi(0)}{\sqrt{|p - e \Phi(0)|^2 + m^2}},$$

equivalently

$$\frac{mv}{\sqrt{1 - |v|^2}} = -e \Phi(0) + p,$$

and

$$p = p(\Phi, \Psi) := \Pi + \frac{1}{4\pi} \langle \Psi, \nabla \Phi \rangle.$$

Thus we have that, at any fixed total momentum Π , we can solve the equations for the fields Φ and Ψ alone, and then recover the particle dynamics by $\dot{q} = v(\Phi, \Psi)$.

Due to the singularity produced by the Dirac mass δ_q , the above reasoning is definitively not rigorous since A is singular at the particle position q (equivalently Φ is singular at the origin). However Example 3 suggests the definition of a well-defined nonlinear operator candidate to describe, in a rigorous way, the classical electrodynamics of a point particle.

Let us define the infinite dimensional manifold

$$\begin{aligned} \mathcal{M} &:= \{(\Phi, \Psi) : \Phi = \Phi_0 + e M v \mathcal{G}^v, \Psi = \Psi_0 - e M v v \cdot \nabla \mathcal{G}^v, \\ &\quad (\Phi_0, \Psi_0, v) \in \bar{H}_*^1 \times L_*^2 \times \mathbb{R}^3, |v| < 1\}, \end{aligned}$$

where the subscript $*$ means “divergenceless”, H^1 and L^2 are defined as in Example 3 but now refer to \mathbb{R}^3 -valued vector fields, and

$$\mathcal{G}^v(x) := \frac{1}{\sqrt{|x|^2 - |v \wedge x|^2}}.$$

Note that

$$A_{LW}^v(t, x) := e M v \mathcal{G}^v(x - vt)$$

satisfies

$$\square A_{LW}^v = 4\pi e M v \delta_q, \quad q(t) = vt,$$

i.e. A_{LW}^v is the Liénard-Wiechert potential corresponding to a particle with constant velocity v .

We identify $T_{(\Phi, \Psi)}\mathcal{M}$, the tangent space of \mathcal{M} at

$$(\Phi, \Psi) \equiv (\Phi_0 + e M v \mathcal{G}^v, \Psi_0 - e M v v \cdot \nabla \mathcal{G}^v),$$

with the Hilbert space

$$\begin{aligned} \mathcal{H}_v &:= \{(\tilde{\Phi}, \tilde{\Psi}) : \tilde{\Phi} = \tilde{\Phi}_0 + e M \tilde{v} \mathcal{G}^v, \tilde{\Psi} = \tilde{\Psi}_0 - e M \tilde{v} v \cdot \nabla \mathcal{G}^v, \\ &\quad \tilde{\Phi} \in L_*^2, \quad (\tilde{\Phi}_0, \tilde{\Psi}_0, \tilde{v}) \in \bar{H}_*^1 \times L_*^2 \times \mathbb{R}^3\} \\ &\simeq \{(\tilde{\Phi}, \tilde{\Psi}) : \tilde{\Phi} = \tilde{\Phi}_\lambda + e M \tilde{v} \mathcal{G}_\lambda^v, \tilde{\Psi} = \tilde{\Psi}_\lambda - e M \tilde{v} v \cdot \nabla \mathcal{G}_\lambda^v, \\ &\quad (\tilde{\Phi}_\lambda, \tilde{\Psi}_\lambda, \tilde{v}) \in H_*^1 \times L_*^2 \times \mathbb{R}^3\}. \end{aligned}$$

Then we define the nonlinear vector field

$$X_e : D(W_e) \subset \mathcal{M} \rightarrow T\mathcal{M}, \quad X_e(\Phi, \Psi) := ((\Phi, \Psi), W_e(\Phi, \Psi))$$

by

$$\begin{aligned} D(W_e) &:= \{(\Phi, \Psi) \in \mathcal{M} : \Phi_0 = \Phi_\lambda + 2e M w v \cdot \nabla \mathcal{G}^v * \mathcal{G}_\lambda^v, \\ &\quad \Psi_0 = \Psi_\lambda + e M w (\mathcal{G}_\lambda^v - 2(v \cdot \nabla)^2 \mathcal{G}^v * \mathcal{G}_\lambda^v), \\ &\quad \Phi_\lambda \in \bar{H}_*^2, \Psi_\lambda \in H_*^1, w \in \mathbb{R}^3, \\ &\quad v = v(\Phi, \Psi) = \frac{p - e \langle \Phi_0 \rangle}{\sqrt{|p - e \langle \Phi_0 \rangle|^2 + m^2}}, \\ &\quad p = p(\Phi, \Psi) := \Pi + \frac{1}{4\pi} \langle \Psi_0, \nabla \Phi_0 \rangle \}, \\ W_e(\Phi, \Psi) &:= (v \cdot \nabla \Phi_0 + \Psi_0, v \cdot \nabla \Psi_0 + \Delta \Phi_0) \\ &\equiv (v \cdot \nabla \Phi + \Psi, v \cdot \nabla \Psi + \Delta \Phi + 4\pi e M v \delta_0). \end{aligned}$$

Here

$$\begin{aligned} F\mathcal{G}_\lambda^v(k) &:= \frac{4\pi}{(2\pi)^{3/2}} \frac{1}{|k|^2 + (iv \cdot k + \lambda)^2}, \\ \langle \Phi \rangle &:= \lim_{R \downarrow 0} \langle \Phi \rangle_R, \end{aligned}$$

with $\langle \Phi \rangle_R$ denoting the average of Φ over the sphere of radius R , and λ is an arbitrary positive parameter. We remark that, as it should be clear from the general results given in the previous sections, the parameter λ has simply the role of allowing a convenient decomposition (into “regular” and “singular” components) of the elements in $D(W_e)$, but plays no role in the definition of the action of W_e , which indeed is λ -independent.

It is not difficult to check, by a direct computation, that

$$W_e(\Phi, \Psi) \in \mathcal{H}_{v(\Phi, \Psi)},$$

so that X_e is a vector field on \mathcal{M} in the differential geometric sense as stated above.

Note that W_e coincides with the linear operator W_0 corresponding to the free wave equation on the dense set $\{\Phi \in \bar{H}_*^2 : \Phi(0) = 0\} \times H_*^1$, so that W_e is a nonlinear singular perturbation of the skew-adjoint W_0 .

Once the vector field X_e is defined, the first question to be posed is: does X_e generate a nonlinear flow $F_e(t)$? At the present we have no definitive answer to this question. The results obtained in the linear case (see [10]) suggest to try to write the presumed solution as $F_e(t)((\Phi(0), \Psi(0))) \equiv (\Phi(t), \Psi(t)) = (\Phi_v(t), \Psi_v(t))$ where $v = v(t)$ is a pre-assigned time-dependent vector and $(\Phi_v(t), \Psi_v(t))$ is the solution of the linear inhomogeneous, time-dependent, wave equation

$$\begin{aligned}\dot{\Phi}_v(t) &= v(t) \cdot \nabla \Phi_v(t) + \Psi_v(t) \\ \dot{\Psi}_v(t) &= v(t) \cdot \nabla \Psi_v(t) + \Delta \Phi_v(t) + 4\pi eM v(t) \delta_0\end{aligned}$$

with initial data $(\Phi(0), \Psi(0)) \in D(W_e)$, and then looking for the right differential equation to be satisfied by $v(t)$ in order that the fields $\Phi_v(t)$ and $\Psi_v(t)$ belong to $D(W_e)$ for any t (and hence fit the correct nonlinear boundary conditions).

6. APPENDIX: SKEW-ADJOINT EXTENSIONS OF SKEW-SYMMETRIC OPERATORS

Let

$$W : D(W) \subseteq \mathcal{H} \rightarrow \mathcal{H}$$

be a skew-adjoint operator on the Hilbert space \mathcal{H} with scalar product $\langle \cdot, \cdot \rangle$ and corresponding norm $\|\cdot\|$. The linear subspace $D(W)$ inherits a Banach space structure by introducing the graph norm

$$\|\phi\|_W^2 := \|\phi\|^2 + \|W\phi\|^2.$$

Thus, for any $\lambda \in \mathbb{R}$, $\lambda \neq 0$, $(-W + \lambda)^{-1} \in \mathcal{B}(\mathcal{H}, D(W))$.

We consider now a linear operator

$$L : D(L) \subseteq \mathcal{H} \rightarrow \mathfrak{h},$$

\mathfrak{h} a Hilbert space, such that:

A.1)

$$D(W) \subseteq D(L) \quad \text{and} \quad L_0 := L|_{D(W)} \in \mathcal{B}(D(W), \mathfrak{h});$$

A.2)

$$\text{Ran}(L_0) = \mathfrak{h};$$

A.3)

$$\overline{\text{Ker}(L_0)} = \mathcal{H}.$$

By A.3 $W_0 := W|_{\text{Ker}(L_0)}$ is a closed densely defined skew-symmetric operator. We want now to define a skew-adjoint extension $\hat{W} \neq W$ of W_0 . It will be a singular perturbation of W since it will differ from W only on the complement of the dense set $\text{Ker}(L_0)$.

We define, for any $\lambda \in \mathbb{R}$, $\lambda \neq 0$, the following bounded operators:

$$\check{G}(\lambda) := L(-W + \lambda)^{-1} : \mathcal{H} \rightarrow \mathfrak{h}, \quad G(\lambda) := -\check{G}(-\lambda)^* : \mathfrak{h} \rightarrow \mathcal{H}.$$

By the first resolvent identity one easily obtains the following (see [11], Lemma 2.1)

Lemma 6.1. *For any $\lambda \neq 0$ and $\mu \neq 0$ one has*

$$\begin{aligned} (\lambda - \mu) \check{G}(\mu)(-W + \lambda)^{-1} &= \check{G}(\mu) - \check{G}(\lambda) \\ (\lambda - \mu)(-W + \mu)^{-1}G(\lambda) &= G(\mu) - G(\lambda). \end{aligned}$$

By [13], Lemma 2.1, and A.2 one has that A.3 is equivalent to A.3)

$$\text{Ran}(G(\lambda)) \cap D(W) = \{0\}.$$

We further suppose that A.4)

$$\text{Ran}(G(\lambda)) \subseteq D(L) \quad \text{and} \quad LG(\lambda) \in \mathcal{B}(\mathfrak{h}).$$

Thus we can define $\Gamma(\lambda) \in \mathcal{B}(\mathfrak{h})$ by

$$\Gamma(\lambda) := -LG(\lambda)$$

and we suppose that A.5)

$$\Gamma(\lambda)^* = -\Gamma(-\lambda).$$

By lemma 6.1 one has

$$(6.1) \quad \Gamma(\lambda) - \Gamma(\mu) = -L_0(G(\lambda) - G(\mu)) = (\lambda - \mu) \check{G}(\mu)G(\lambda)$$

and thus, by A.5 and [11], Proposition 2.1, the operator $\Gamma(\lambda)$ is boundedly invertible for any real $\lambda \neq 0$.

Theorem 6.2. *For any real $\lambda \neq 0$, under the hypotheses A.1-A.5, the bounded linear operator*

$$(-W + \lambda)^{-1} + G(\lambda)\Gamma(\lambda)^{-1}\check{G}(\lambda)$$

is a resolvent of a skew-adjoint operator \hat{W} such that

$$\left\{ \phi \in D(\hat{W}) \cap D(W) : \hat{W}\phi = W\phi \right\} = \text{Ker}(L_0).$$

It is defined by

$$\begin{aligned} D(\hat{W}) &:= \left\{ \phi \in \mathcal{H} : \phi = \phi_\lambda + G(\lambda)\Gamma(\lambda)^{-1}L_0\phi_\lambda, \quad \phi_\lambda \in D(W) \right\}, \\ (-\hat{W} + \lambda)\phi &:= (-W + \lambda)\phi_\lambda. \end{aligned}$$

Such a definition is λ -independent and the decomposition of ϕ entering in the definition of the domain is unique.

Proof. By (6.1) $\hat{R}(\lambda) := (-W + \lambda)^{-1} + G(\lambda)\Gamma(\lambda)^{-1}\check{G}(\lambda)$ satisfies the resolvent identity $(\lambda - \mu)\hat{R}(\mu)\hat{R}(\lambda) = \hat{R}(\mu) - \hat{R}(\lambda)$ (see [11], page 115, for the explicit computation) and, by A.5, $\hat{R}(\lambda)^* = -\hat{R}(-\lambda)$. Moreover, by A.3, $\hat{R}(\lambda)$ is injective. Thus $\hat{W} := -\hat{R}(\lambda)^{-1} + \lambda$ is well-defined on $D(\hat{W}) := \text{Ran}(R(\lambda))$, is λ -independent and is skew-symmetric. It is skew-adjoint since $\text{Ran}(W \pm \lambda) = \mathcal{H}$ by construction. \square

REFERENCES

- [1] S. Albeverio, F. Gesztesy, R. Høegh-Krohn, H. Holden: *Solvable Models in Quantum Mechanics*. Berlin, Heidelberg, New York: Springer-Verlag 1988
- [2] S. Albeverio, P. Kurasov: *Singular Perturbations of Differential Operators*. Cambridge: Cambridge Univ. Press 2000
- [3] A. Alonso, B. Simon: The Birman-Kreĭn-Vishik Theory of Self-Adjoint Extensions of Semi-Bounded Operators. *J. Operator Theory* **4** (1980), 251-270
- [4] M. Bertini, D. Noja, A. Posilicano: Wave Equations with Point Interactions in Finite Energy Spaces. *J. Math. Phys.* **42** (2001), 2184-2202
- [5] M. Bertini, D. Noja, A. Posilicano: Rigorous Dynamics and Radiation Theory for a Pauli-Fierz Model in the Ultraviolet Limit. Preprint 2004, available at arXiv.org/abs/math-ph/0407002
- [6] J.A. Goldstein: *Semigroups of Linear Operators and Applications*. New York: Oxford Univ. Press 1985
- [7] T. Kato: Scattering Theory with Two Hilbert Spaces. *J. Funct. Anal.* **1** (1967), 342-369
- [8] V. Koshmanenko: *Singular Quadratic Forms in Perturbation Theory*. Dordrecht: Kluwer Acad. Publishers 1999
- [9] V. Kostykin, R. Schrader: Kirchhoff's Rule for Quantum Wires. *J. Phys. A: Math. Gen.* **32** (1999), 595-630
- [10] D. Noja, A. Posilicano: On the Point Limit of the Pauli-Fierz Model. *Ann. Inst. H. Poincaré Phys. Theor.* **71** (1999), 425-457
- [11] A. Posilicano: A Kreĭn-like Formula for Singular Perturbations of Self-Adjoint Operators and Applications. *J. Funct. Anal.* **183** (2001), 109-147
- [12] A. Posilicano: Boundary Conditions for Singular Perturbations of Self-Adjoint Operators. *Oper. Theory Adv. Appl.* **132** (2002), 333-346
- [13] A. Posilicano: Self-Adjoint Extensions by Additive Perturbations. *Ann. Scuola Norm. Sup. Pisa Cl. Sci.(5)* **II** (2003), 1-20
- [14] B. Weiss: Abstract Vibrating Systems. *J. Math. Mech.* **17** (1967), 241-255

DIPARTIMENTO DI SCIENZE FISICHE E MATEMATICHE, UNIVERSITÀ DELL'INSUBRIA,
I-22100 COMO, ITALY

E-mail address: posilicano@uninsubria.it